United States
Environmental Protection
Agency

Research and Development

Environmental Monitoring and Support
Laboratory
Cincinnati OH 45268

EPA-600/S4-83-021 Aug. 1983

EPA

Project Summary

AMENDED
SEE PAGES 1, 2, AND 3

Mathematical Models Associated with Point and Line Source Discharges in Rivers

Philip C. L. Lin

A literature search was conducted in the area of mass dispersion from instantaneous and continuous sources in bounded and unbounded flows. This revealed that most of the studies are concentrated on flows in channels with simple geometries such as constant depth and width, and with constant fluid velocity and dispersion coefficients. Very often the depth-averaged value of concentration, rather than point value, is assumed to approximate the complicated nature of mixing, especially in bounded flows in natural rivers. Analytical solutions often require tedious and repetitious calculations. This study presents a mathematical model consisting of only two nondimensionalized variables capable of generating universal concentration profiles of the pollutants in a rectangular channel flow for specified initial and boundary conditions. Determination of diffusion coefficients, mixing distance, sampling location, and number of sampling points in a cross-section is possible.

In other aspects, determination of the minimum sampling frequency required to collect a representative composite sample based on the previous effort is reexamined. It is concluded that a minimum sampling frequency of eight (8) for most of the flow and concentration patterns in the complete mixing zone is sufficient to provide enough accuracy for a representative sample collected by three out of four known compositing techniques.

This Project Summary was developed by EPA's Environmental Monitoring and Support Laboratory, Cincinnati, OH, to announce key findings of the research project that is fully documented in a separate report of the same title (see Project Report ordering information at back).

Introduction

When pollutants are discharged into a river, it is often desirable to be able to predict their dispersion downstream from the release point. This is because of the importance of obtaining representative samples for any monitoring or enforcement program. Hence, the more accurately one can predict the behavior of the pollutant dispersion, the more representative a sample one can expect to collect.

For the prediction of the pollutant dispersion, one must know the hydraulics of the stream, the physical, chemical, or biological change, and the rate of the pollutant mixing. Because of the complexities of variations of depth and width in most natural rivers, fluid velocity distribution, and the rate of mixing may vary from one location to another. It is, therefore, difficult to fully describe the mixing phenomenon in natural rivers. The approach generally used to predict pollutant mixing in a stream is based on assumptions that an average velocity, an average depth and even a constant width for the river are adopted to characterize the flow in an irregularly shaped water channel. Many unknown aspects of the mixing in the flow are then implicitly lumped into coefficients of dispersion. A review of the literature reveals that most of the analytical solutions of pollutant mixing are either for instantaneous point and line sources in an unbounded fluid or for continuous line sources in both unbounded and bounded fluid. Among the two categories of sources, the instantaneous sources are only of theoretical interest in a practical sense. The continuous sources, however, provide a deeper insight into the behavior of the mass being transported, as actual pollutant spills are never instantaneous, but are spread over a finite time interval. Although analytical solutions for predicting pollutant mixing are closed forms, they often require tedious and time-consuming calculations. Furthermore, repetitious calculations are needed for any change of values in variables such as velocity, depth, or dispersion coefficients. Most of the solutions also have the aforementioned restrictions such as a constant channel width and constant velocity.

It is the intention in this study to develop a mathematical model which combines variables such as distance, dispersion coefficients, depth, and kinematic viscosity into groups of nondimensionalized variables and, therefore, to produce universal concentration profiles for the river pollutants under specified conditions in the area of mass dispersion. This model also has its limitations and advantages. It is restricted to a rectangular channel flow, but, however, is allowed to have a variable velocity profile across a section of the channel.

Also included in this study is a further examination of the minimum sampling frequency for collecting representative samples under completely mixed conditions for variable flow and concentration patterns.

Discussion

Most predictions of concentration distributions from point or line sources in bounded or unbounded flows are expressed as closed-form solutions under conditions of uniform or average flow velocity, constant channel width or dispersion coefficients. Only those solutions from continuous sources in bounded flows seem applicable to simulate problems in rivers. Conclusions from the selected literature review in this area may be summarized as follows:

- *For flow in a rectangular channel with constant velocity and constant dispersion coefficient in the transverse direction of the flow*
  Eqs. (1) and (2) may be used to evaluate the concentration distributions at any location of the river. Eq. (3) may be used to determine the mixing distance where the material being transported in the channel is completely mixed.

$$C(x,z) = \frac{Q_m}{h(\pi e_z u x)^{1/2}} \sum_{n=-\infty}^{n=\infty} \exp \left\{ - \frac{u(z-2nH)^2}{4 e_z x} \right\} \qquad (1)$$

$$C(x,z) = \frac{Q_m}{hHu} + \frac{2Q_m}{hHu} \sum_{n=1}^{n=\alpha} \frac{\cos\frac{n\pi z}{H}}{(1+\alpha_n)^{1/2}} \exp \left\{ \frac{ux}{2e_z} \left(1-(1+\alpha_n)^{1/2}\right) \right\} \qquad (2)$$

$$X_m = \frac{0.445 H^2 u}{e_z} \qquad (3)$$

where

C = concentration of material being transported in the channel

X = coordinate in the flow direction in the channel

z = coordinate in the transverse direction in the channel

Q_m = a constant release of material emanating at the origin of the coordinates

h = depth of the channel

e_z = dispersion coefficient in the transverse direction, z, in the channel

u = constant flow velocity

H = constant channel width

$$\alpha_n = \left(\frac{2 e_z n \pi}{uH} \right)^2$$

X_m = mixing distance

- *For flow in a river with variable depth and width*

Eq. (4) can be used to calculate the concentration distributions if the distributions of flow velocity, river depth and dispersion coefficients across the river at any section are known.

$$C'(\alpha,q') = \frac{1}{2(q'_{s2}-q'_{s1})} \Big[\sum_{n=o}^{n=\infty} \Big\{ \mathrm{erf}\, \frac{\alpha(q'_{s2}+2n-q')}{\sqrt{2}} - \mathrm{erf}\, \frac{\alpha(q'_{s1}+2n-q')}{\sqrt{2}} + \mathrm{erf}\, \frac{\alpha(q'_{s2}+2n+q')}{\sqrt{2}} - \mathrm{erf}\, \frac{\alpha(q'_{s1}+2n+q')}{\sqrt{2}} \Big\}$$

$$+ \sum_{n=0}^{n=\infty} \Big\{ \mathrm{erf}\, \frac{\alpha(q'_{s2}-2n-q')}{\sqrt{2}} - \mathrm{erf}\, \frac{\alpha(q'_{s1}-2n-q')}{\sqrt{2}} + \mathrm{erf}\, \frac{\alpha(q'_{s2}-2n+q')}{\sqrt{2}} - \mathrm{erf}\, \frac{\alpha(q'_{s1}-2n+q')}{\sqrt{2}} \Big\} \Big] \qquad (4$$

where

$$\mathrm{erf}\, q = \frac{2}{\sqrt{\pi}} \int_0^q e^{-p^2}\, dp$$

$$\alpha = \frac{Q^2}{2x\, e_z u h^2}$$

$$q' = \frac{q}{Q}$$

$$C' = \frac{C}{\frac{1}{Q}\int_0^Q C\, dq}$$

$$q = \int_0^z uh\, dz$$

$$Q = \int_0^H uh\, dq$$

q'_{s1}, q'_{s2} = the two end points (the line source e) pressed as fraction of the total discharge

This study presents simple universal concentration profiles for a rectangular channel flow. The profiles are valid for steady state condition from continuous sources in the steam. The applications for this model can be summarized as follows:

- *Determination of Schmidt number, S_z, and dispersion coefficient, e_z*

  The concentration distributions across a water channel at any location x are measured and also are obtained from the model. Comparison of the results will indicate that an appropriate value of nondimensionalized distance X* should be selected. Therefore, Schmidt number, S_z, and dispersion coefficient, e_z can be determined as follows:

$$S_z = \frac{X}{X^* \mathrm{Re} H} \tag{5}$$

where

$\mathrm{Re} = \frac{1.5\ UH}{\nu}$, Reynolds number

ν = kinematic viscosity of water
U = average velocity in the channel
H = channel width

$$e_z = \frac{\nu}{S_z}$$

- *Determination of sampling location X*

  Once the Schmidt number is known, one can choose a known concentration distribution of X* and locate the sampling location X as follows:

$$X = X^*\ \mathrm{Re}\ S_z\ H \tag{6}$$

- *Determination of number of sampling points at location X*

  Since the concentration distributions are known at a certain value of X*, the value of X can be calculated. The number of sampling points are then determined from the known concentration distributions by Simpson's rule or any other approximate technique.

- *Determination of the mixing distance X_m*

  The mixing distance for a continuous discharge in a rectangular channel is

$$X_m = K\ \mathrm{Re} S_z H \tag{7}$$

where

K = 0.30 for a continuous discharge at one of the banks
K = 0.14 for a continuous discharge in the center fo the channel.

Although this effort is restricted to a rectangular channel flow, it also provides an approximate solution for channel flows with constant width and moderate variation of depth. This study is a continuous effort in this area and will be helpful to field personnel in selecting the sampling location and number of sampling points.

In the study of sampling frequency, it is found that a minimum sampling frequency of m = 8 is sufficient to provide enough accuracy for samples being collected for most of the flow and concentration patterns studied.

Conclusions

The capability to analytically solve the mixing phenomenon in rivers is very limited, as can be seen from the literature being reviewed. The present mathematical model which includes velocity variations also has its limitations such as the restrictions of constant channel width and depth. There is, therefore, a need to develop a model which takes variations of depth, width, and velocity in a river into consideration. The possible solution may only be numerically obtained by a computer.

*The EPA author **Philip C. L. Lin** is with the Environmental Monitoring and Support Laboratory, Cincinnati, OH 45268.*

The complete report, entitled "Mathematical Models Associated with Point and Line Source Discharges in Rivers," (Order No. PB 83-207 373; Cost: $10.00, subject to change) will be available only from:
National Technical Information Service
5285 Port Royal Road
Springfield, VA 22161
Telephone: 703-487-4650

The EPA author can be contacted at:
Environmental Monitoring and Support Laboratory
U.S. Environmental Protection Agency
Cincinnati, OH 45268

☆U.S GOVERNMENT PRINTING OFFICE 1983–659-017/7224

United States
Environmental Protection
Agency

Center for Environmental Research
Information
Cincinnati OH 45268