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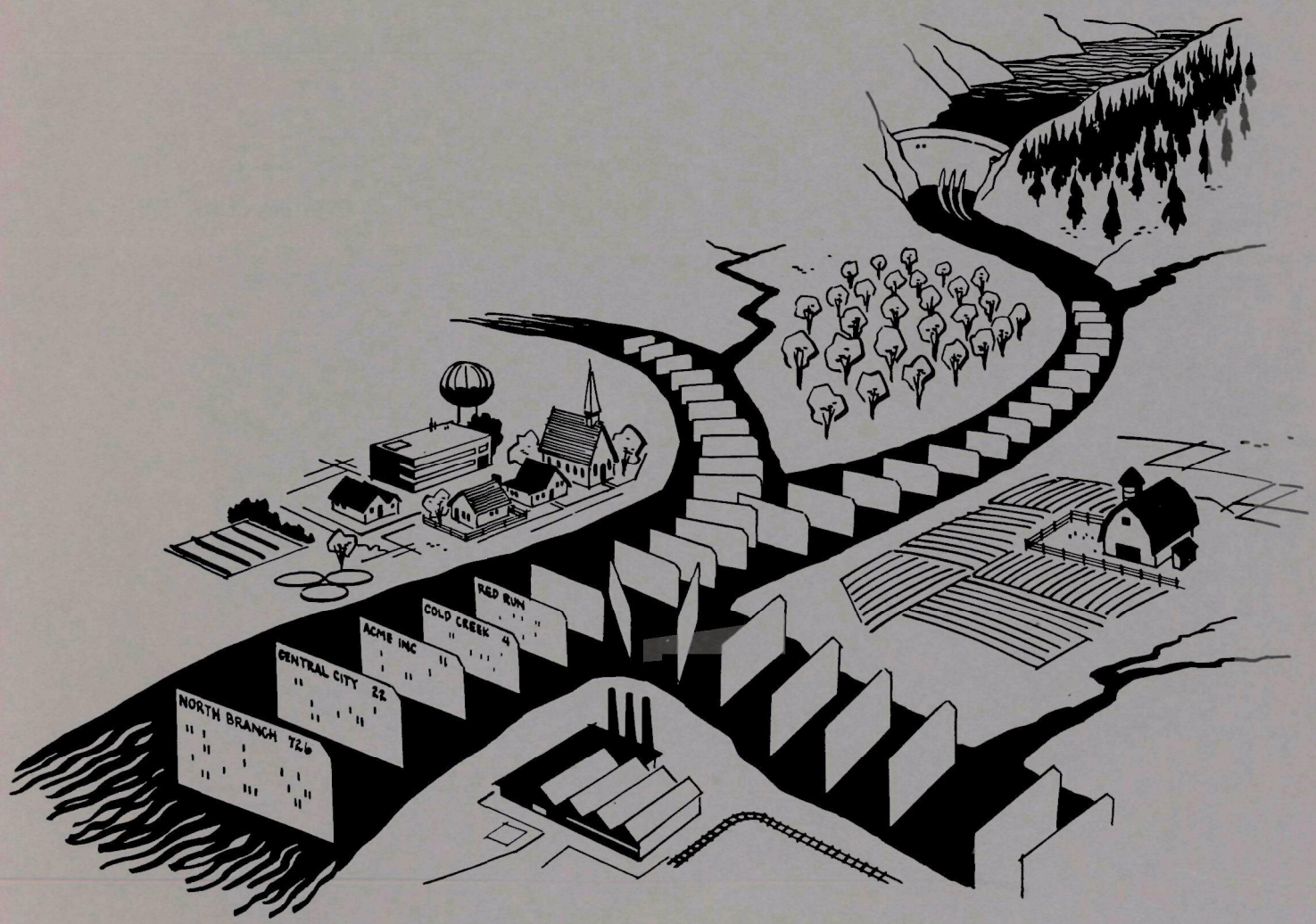
EXTENSIONS OF MATHEMATICAL PROGRAMMING FOR REGIONAL WATER QUALITY MANAGEMENT



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EXTENSIONS OF MATHEMATICAL PROGRAMMING FOR REGIONAL WATER QUALITY MANAGEMENT

by

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for the

OFFICE OF RESEARCH AND MONITORING

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ABSTRACT

This report extends the work contained in the earlier work, "Mathematical Programming for Regional Water Quality Management." A mixed integer, continuous variable non-linear programming model is developed which promises to be much more realistic and effective in selecting an optimal configuration of regional treatment plants and pipe juncture nodes. A new Bender's type decomposition for the non-linear model is also presented as an appropriate and feasible solution technique. In the area of the physical response of the environment, the earlier work linked the dissolved oxygen profile and the control options through a constant transfer matrix obtained as a steady state solution to a system of differential equations. This work shows how the transfer coefficients can be recalculated periodically in the course of the system optimization. Full scale computations are carried out demonstrating this capability. The theoretical framework for including two additional treatment methods, mechanical re-aeration and flow augmentation via reservoirs is also developed. In the social and economic area, a new proposal for paying for a regional system is presented. Illustrative computations using the new proposal are included. This report was submitted in fulfillment of project 16110EGQ under the partial sponsorship of the Environmental Protection Agency.

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I. CONCLUSIONS AND RECOMMENDATIONS

The benefits to be derived from the use of regional optimization models for water quality management would seem to be enormous. Regional optimization models are applicable to bays, estuaries and rivers. They are capable of dealing with any quality goal for which an appropriate system of diffusion differential equations can be obtained to represent the physical response of the environment. In particular, they are capable of dealing with dissolved oxygen, salinity and thermal contamination. The control options include treatment at the source, by-pass piping, regional treatment plants, mechanical re-aeration and flow maintenance augmentation. The output from the model gives the least cost solutions to achieve the pre-selected goals using the full range of control options. The optimal solutions would specify the treatment levels at the major sources of contamination, the complete regional pipe network, the location and levels of treatment of an optimal number of regional plants, the location and level of mechanical re-aeration and the location and level of flow augmentation. The model could be used in conjunction with regional growth of demand projections to plan the long term treatment capacity expansion. It could ultimately be used in a real time regional control process utilizing physical and chemical treatment as well as the traditional biological treatment to capitalize on seasonal and daily flucuation to greatly reduce yearly and daily treatment costs. This report develops the theoretical possibility of including two additional treatment methods, mechanical re-aeration and flow maintenance or augmentation via reservoirs. In order to deal with the question of the effect on the re-aeration coefficient of radical shifts in flow, Section VI investigates the effect of flow on the depth and velocity of a river. Finally, Section VII takes a preliminary look at the transient effects in an estuary as opposed to the long term steady state conditions.

In the area of system optimization, Section IV examines the question of an optimal configuration of regional treatment plants and pipe junction points. The original model deals with this question by forcing a solution using only the regional plants. This forces the plants to operate at sufficiently high levels as to utilize their economies of scale. When the other treatment methods are then introduced, they are compelled to compete against the efficiently operating plants. Where economically justified they then force out and close the non-competitive plants. This is computationally wasteful as the plants must first be opened then some of them closed. It is also mathematically doubtful as to the global optimality of this procedure. The mathematical foundations of a potentially much more efficient procedure using an integer non-linear programming model are presented. Computational effectiveness hinges upon the power of a new decomposition technique and the capability of rapidly solving integer programming problems.

the realism of the models would seem to be adequate at this time although they may ultimately be considereably improved. The computational feasibility for full scale problems has been demonstrated. It would seem entirely practical to proceed to actual pilot studies at this time with the tools already at hand. Further development of the theoretical possibilities should proceed concomitant with the pilot studies. In particular computer implementation of the new mixed integer, continuous variable non-linear model should be undertaken to substantiate its potential value. The possibility of including the additional treatment methods of mechanical re-aeration and flow augmentation should also be exploited. On a longer horizon the dynamic or transient behavior of the physical environment should be studied more closely. It would be entirely possible to realize substantial gains from an optimal holding and discharge policy based on the transient behavior of the environment. Finally the mathematical programming model for regional water quality management should be coupled with a regional demand model. Rational planning over a realistic amortization period requires fairly good estimates of projected demand over the period. The capital budgeting problem associated with the optimal timing and amount of treatment capacity expansion should also be carefully studied.

II. INTRODUCTION

This report continues and extends the work of the earlier report, "Mathematical Programming for Regional Water Quality Management." The goal of this work is to develop a general practical planning tool to provide optimal solutions for the complex choices involved in balancing alternative methods for attaining water quality goals. The work encompasses three broad related areas. The first area is the social and economic question of what are desirable and feasible water quality goals and what is the best way to achieve and pay for them. The second area is the complex technical engineering problem of the physical response of the environment to various control strategies. The third area is the mathematical problem of obtaining least cost solutions in the total regional water system that will satisfy the desired goals.

In the area of physical response our earlier work presented a general mathematical model relating the control options of by-pass piping, treatment at regional treatment plants and treatment at the source to the quality goal specified as a dissolved oxygen profile on an estuary. The original model was limited in several respects. The coupling of the dissolved oxygen profile goal and the control options was through a constant transfer matrix obtained as a steady state solution to a system of differential equations first given by Thomann [10]. This is inadequate when by-pass piping and regional treatment plants substantially alter the flow. Section IV of this report re-examines the question of the transfer functions. This work shows how the transfer coefficients can be re-calculated periodically in the course of the system optimization. It further develops the theoretical possibility of including two additional treatment methods, mechanical re-aeration and flow augmentation via reservoirs. In order to deal with the question of the effect on the re-aeration coefficient of radical shifts in flow, Section V investigates the effect of flow on the depth and velocity of a river. Finally, Section VI takes a preliminary look at the transient effects in an estuary as opposed to the long term steady state conditions.

In the area of system optimization, Section III examines the question of an optimal configuration of regional treatment plants and pipe junction points. The original model deals with this question by forcing a solution using only the regional plants. This forces the plants to operate at sufficiently high levels as to utilize their economies of scale. When the other treatment methods are then introduced, they are compelled to compete against the efficiently operating plants. Where economically justified, they then force out and close the non-competitive plants. This is computationally wasteful as the plants must first be opened then some of them closed. It is also mathematically doubtful as to the global optimality of this procedure. The mathematical foundations

of a potentially much more efficient procedure using an integer non-linear programming model are presented. Computational effectiveness hinges upon the power of a new decomposition technique and the capability of rapidly solving integer programming problems.

In the social and economic area, Section VII discusses economic policy for regional water quality control. A new proposal for paying for a regional system is presented. Illustrative computations using the new proposal are included.

III. SELECTION OF A WASTE TREATMENT PLANT CONFIGURATION

In our earlier report, "Mathematical Programming for Regional Water Quality Management," the model presented dealt with the difficult question of the best treatment plant configuration in a very limited manner. Flow of effluent was restricted flow direct to a discharge point or flow to a single plant and then from the plant to a discharge point. There was no flow between plants and hence no way to achieve say primary treatment for a large percentage of the effluent at huge, cheap primary plants and subsequent treatment for a much smaller percentage at smaller more expensive secondary or tertiary plants. Since pipe juncture nodes can be considered plants without treatment, the flow pattern was restricted to single juncture points. This resulted in practice in manual pipe consolidation and weakened results.

The solution procedure for the original model required forcing a solution using only the regional plants. This forces the plants to operate at sufficiently high levels as to utilize their economies of scale. When the other treatment methods are then introduced, they are compelled to compete against the efficiently operating plants. Where economically justified they then force out and close the non-competitive plants. This is computationally wasteful as the plants must first be opened and driven down and closed. There are also difficult problems with infinite derivatives in the cost function with the initial very low or zero flows. These problems can be handled much more effectively using discrete integer variables in the formulation of the model. This possibility was not exploited originally becuase of the limitations of the solution techniques for large mixed intefer and continuous variable non-linear programming problems. Recent research has greatly expanded our capability to solve integer programming problems. This extended capability coupled with new and more powerful decomposition techniques change the picture radically. It now seems attractive to switch to the mixed integer, continuous variable formulation.

Mixed Integer Regional Treatment Model

In this model we can use a much more extensive and realistic supporting flow network. This is illustrated by the schematic shown in Figure 1 on the following page.

The nodes in the network are of three types. A node is a polluter or source, a sink or a discharge point, or an intermediate node in the network consisting of a pipe junction or treatment plant. The sources are fixed at the present sites of the identified major polluters. The discharge nodes are the presently used discharge sections of the estuary plus any additional potentially attractive discharge sections. The intermediate nodes consist of any existing treatment plants or pipe junctures plus any additional potentially attractive sites. Combining existing

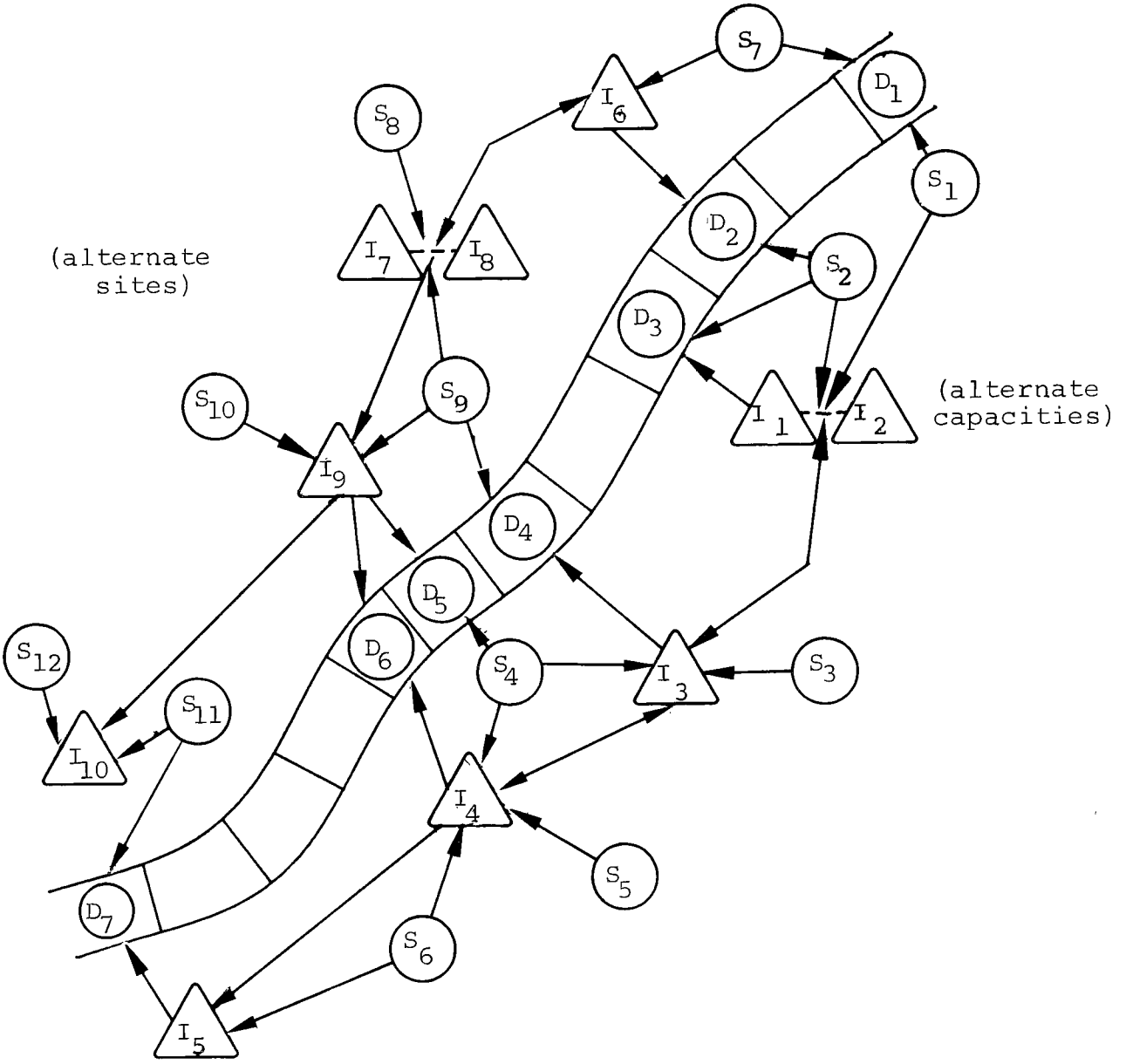


FIGURE 1

SCHEMATIC OF GENERAL SYSTEM

and new sites into an optimal total network is particularly nice in this approach because the fixed or capital costs can be separated from the variable or operating costs. This permits a true comparison of the marginal cost of a new plant to an existing plant. The arcs in the network are pipes. They can consist of existing pipes and potentially attractive additional pipes. Again the fixed and variable costs can be separated. Also it is possible to have discrete alternate capacities and enforce a choice of pipe size. Similarly, we can enforce a

choice between alternate plant capacities at a given site or between alternate plant sites.

In the mathematical statement of the model we use the following:

Notation

Physical Quantities

A - matrix of transfer coefficients (mg/ℓ per lb./day)

C - vector of D. O. goals (mg/ℓ)

r_i - % removal of B.O.D. at node i

q_{ij} - flow from node i to node j (MGD)

J_i - concentration leaving node i (lb/MG)

m_i - mass of B.O.D. leaving node i (lb/D)

Costs

f_i - capital cost of plant i of a given capacity ($)

v_i - operating costs of plant i ($/MG)/% removal)

f_{ij} - capital cost of pipe between node i and node j of given capacity ($)

v_{ij} - operating cost of pipe between node i and node j of given capacity ($/MG)

p_i - cost of additional treatment at source ($/% removal)

Index Sets

$R = \{j \mid j \text{ is a section or reach of the estuary}\}$

$S = \{i \mid \text{node } i \text{ is a source}\}$

$D = \{i \mid \text{node } i \text{ is a discharge point}\}$

$I = \{i \mid \text{node } i \text{ not a source or sink}\}$

$N = S \cup D \cup I$ set of nodes; $a \subset N \times N$ set of arcs

$a(x) = \{y \in N \mid (x,y) \in a\}$ nodes "after" x

$b(x) = \{y \in N \mid (y,x) \in a\}$ nodes "before" x

Integer Variables

$$z_i = \begin{cases} 0 & \text{plant } i \text{ is closed} \\ 1 & \text{plant } i \text{ is open} \end{cases}$$

$$z_{ij} = \begin{cases} 0 & \text{pipe } (i,j) \text{ is closed} \\ 1 & \text{pipe } (i,j) \text{ is open} \end{cases}$$

The mathematical model is:

Subject to

Conservation of Flow

$$\sum_{j \in a(i)} q_{ij} - \sum_{k \in b(i)} q_{ki} = 0 \qquad i \in I$$

$$\sum_{j \in a(i)} q_{ij} = \bar{q}_i \qquad i \in S \quad (\bar{q} \text{ current discharge MG/D})$$

Conservation of Concentration

$$J_i = \frac{\left(\sum_{k \in b(i)} q_{ki} J_k\right) \cdot (1 - r_i)}{\left(\sum_{k \in b(i)} q_{ki}\right)} \qquad i \in I \cup D$$

$$J_i = \frac{\bar{m}_i}{\bar{q}_i} (1 - r_i) \qquad i \in S \quad (\bar{m}_i \text{ current discharge lb})$$

Capacity at Plant (ℓ_i - lower, μ_i - upper capacity)

$$z_i \cdot \ell_i \leq \sum_{k \in b(i)} q_{ki} \leq z_i \mu_i$$

(Note that when the plant is closed or not chosen for the network all flow is shut off and when it is opened it functions in a restricted range. By breaking up the range non-convexity can be eliminated and more truly global optimum achieved. This would require merely adding some:

Alternative Constraints (Plants)

$$\sum_{t \in E_i} z_t = 1$$

These constraints would force a choice of only one of the plants in the set E_i.)

Capacity of Pipes (ℓ_{ij} - lower, μ_{ij} - upper capacity)

$$z_{ij} \cdot \ell_{ij} \leq q_{ij} \leq z_{ij} \cdot \mu_{ij}$$

(Again the pipes can be eliminated or forced to function in a restricted range. By breaking up the range non-convexity can be eliminated. Again we would employ:

Alternative Constraints (Pipes)

$$\sum_{(t,s)\in E_{ij}} z_{ts} = 1$$

These constraints would force a choice of one of the alternatives in the set E_{ij}.)

The quality goals are enforced by introducing the auxillary variables

$$m_t = J_t \cdot \left(\sum_{k\in b(t)} q_{kt} \right) \quad t \in D$$

$\bar{m}_t$ = present discharge in lb's

and the following:

Quality Constraints

$$\sum_{t\in D} a_{it}(m_t - \bar{m}_t) \leq -c_i \quad i \in R$$

The objective is to minimize the total cost function

minimize

$$T(z_t, z_{ts}, q, r) = PL(z_t, q) + PI(z_{ts}, q) + PO(r)$$

which is broken down into:

Plant Costs

$$PL(z_t, q, r) = \sum_{t\in I} z_t \cdot \left(f_t + v_t(r) \cdot \sum_{k\in b(t)} q_{kt}\right)$$

Pipe Costs

$$PI(z_{ts}, q) = \sum_{(t,s)\in A} z_{ts} \cdot (f_{ts} + v_{ts} q_{ts})$$

Polluter Costs

$$PO(r) = \sum_{t\in S} r_t \cdot p_t$$

Solution Technique

The foregoing complex mixed integer continuous variable non-linear programming model presents a truly formidable computational challenge. Although a direct implicit enumeration of the integer variables coupled with the bounding off effect of solving the remaining non-linear programming problems might be attempted, the outcome of such a venture would be extremely doubtful. Recent advances in Bender's type (see [2]) decomposition offer a much more promising avenue of attack. A Bender's type decomposition applicable to a general non-linear programming problem

Subject to

$$g^i(y_1,y_2) \leq 0$$

$$g^k(y_2) \leq 0$$

$$\min g^m(y_1,y_2)$$

is possible. In this type of decomposition a subset of the variables say the y_2 vector can be used to parametrize a relaxed Sub-Problem

Subject to

$$g^i(y_1 \cdot \bar{y}_2) \leq 0$$

$$\min g^m(y_1,\bar{y}_2)$$

and the parameters y_2 need not be continuous variables.

Whenever a fixed choice of parametes $\bar{y}_2$ satisfying the remaining constraints

$$g^k(\bar{y}_2) \leq 0$$

permits a feasible solution to the sub-problem, we obtain a feasible solution to the complete problem. In order to avoid a random search through plausible choices of $\bar{y}_2$, we employ a class of lower bound functions D. In a stepwise procedure we generate members of this class. By ensuring that at least each lower bound function already generated indicates a better possible solution than the best known incumbent feasible solution we guide the choice of the parameter variables. In point of fact since it is a necessary condition that each lower bound function be less than the incumbent when there exists a better solution, the lower bound functions provide us with a convergence criteria.

In order to generate the class of lower bound functions of interest here we require the following expansions of the functions.

<u>Expansion 1</u> (With respect to y_1)

$$(1) \quad g^i(y_1,y_2) = g^i(y_1^o,y_2) + \nabla g^i(y_1^o,y_2) \cdot \Delta y_1 + \tfrac{1}{2}\Delta y_1^T H^i(\tilde{y}_1,y_2)\Delta y_1$$

Now in order to eliminate the gradients and utilize information furnished by the dual variables in the sub-problems, we must make:

<u>Expansion 2</u> (With respect to y_2)

$$(2) \quad \nabla g^i(y_1^o,y_2) \cdot \Delta y_1 = \nabla g^i(y_1^o,y_2^o) \cdot \Delta y_1 + \Delta y_2^T \bar{H}^i(y_1^o,\tilde{y}_2) \cdot \Delta y_1$$

where

$$\bar{H}^i(y_1^o,\tilde{y}_2) = \frac{\partial^2 g^i(y_1,y_2)}{\partial y_{1k} y_{2j}}$$

is a matrix of mixed partial derivatives. Combining the error terms

$$(3) \quad R^i(\Delta y_1,\Delta y_2) = \tfrac{1}{2}\Delta y_1^T H^i(\tilde{y}_1,y_2) \cdot \Delta y_1 + \Delta y_2^T \bar{H}^i(y_1^o,\tilde{y}_2) \cdot \Delta y_1$$

yields the desired representation of each function

$$(4) \quad g^i(y_1,y_2) = g^i(y_1^o,y_2) + \nabla g^i(y_1^o,y_2^o) \cdot \Delta y_1 + R^i(\Delta y_1,\Delta y_2)$$

Using this representation of our functions and the following relation,

$$(5) \quad \sum_i x_i \nabla g^i(y_1^o,y_2^o) = \nabla g^m(y_1^o,y_2^o)$$

which is derived as equation (14) on page 17 of our previous report, "Mathematical Programming for Regional Water Quality

Management", yields our class of lower bound functions. The argument employed in establishing this class of lower bound functions is similar to the argument used to establish the "Weak Duality Theorem" of linear programming. Any ϵ-feasible solution of the sub-problems satisfies

$$(6) \quad g^i(y_1,y_2) = g^i(y_1^o,y_2) + \nabla g^i(y_1^o,y_2^o)\cdot\Delta y_1 + R^i(\Delta y_1,\Delta y_2) \leq \epsilon_1$$

or

$$(7) \quad \nabla g^i(y_1^o,y_2^o)\cdot\Delta y_1 \leq -g^i(y_1^o,y_2) - R^i(\Delta y_1,\Delta y_2) + \epsilon_1$$

Using (5) and (7) we demonstrate that as a function of y_2,

<u>For any value of y_1</u>

$$\begin{aligned}
(8) \quad g^m(y_1,y_2) &= g^m(y_1^o,y_2) + \nabla g^m(y_1^o,y_2^o)\cdot\Delta y_1 + R^m(\Delta y_1,\Delta y_2) \\
&= g^m(y_1^o,y_2) + [\sum_i x_i \nabla g^i(y_1^o,y_2^o)]\cdot\Delta y_1 + R^m(\Delta y_1,\Delta y_2) \\
&= g^m(y_1^o,y_2) + \sum_i x_i[\nabla g^i(y_1^o,y_2^o)\cdot\Delta y_1] + R^m(\Delta y_1,\Delta y_2) \\
&\geq g^m(y_1^o,y_2) + \sum_i(-x_i)\cdot g^i(y_1^o,y_2) \\
&\quad + \sum_i(-x_i)\cdot R^i(\Delta y_1,\Delta y_2) + R^m(\Delta y_1,\Delta y_2) \\
&\quad + \sum_i x_i\cdot\epsilon_1
\end{aligned}$$

For convenience we restate the conditions for a stationary point of the non-linear algorithm presented in our previous report [4] on page 19.

<u>Local Linear Stationary Point</u>

$\exists$ a set of dual variables $x_p \ni$

1) $\sum_p(-x_p) > B_1$ (insufficient resolution)

$\exists$ a w and H $\ni$

2) $\sum_{p \in H} (-x_p) \cdot g^p(y^o) > -g^w(y^o) - \epsilon$ (Inconsistency)

$\exists$ a $y_o \in F \ni$

3) $\sum_{i=1}^{m-1} (-x_i) \cdot g^i(y^o) > -\epsilon$ (Optimality)

Now choose $\epsilon_1 = \epsilon/B_1$ and assume that condition 1) of a local linear stationary point is not violated then

$$\sum_i x_i \geq -B_1 \quad \text{and} \quad \sum_i x_i \cdot \epsilon_1 \geq -\epsilon$$

and when the error expression

(9) $\sum_i (-x_i) \cdot R^i(\Delta y_1, \Delta y_2) + R^m(\Delta y_1, \Delta y_2) \geq 0$

we have our lower bound function

(10) $g^m(y_1^o, y_2) + \sum_i (-x_i) \cdot g^i(y_1^o, y_2) - \epsilon \leq g^m(y_1, y_2)$

(Note that in the case where the functions are separable in y_1 and y_2 as in our model, the convexity of the sub-problems in the continuous variables is sufficient to ensure condition (9) and the validity of the decomposition.)

The decomposition procedure then consists of alternately solving the relaxed sub-problems and the following

<u>Master Problem</u>

Find a feasible solution to,

1) $g^k(y_2) \leq 0$

2) $g^m(y_1^o, y_2) + \sum_i (-x_i) \cdot g^i(y_1^o, y_2) - \epsilon \leq g^m(y_1^c, y_2^c) - 2\epsilon$ (incumbent)

3) $g^w(y_1^o, y_2) + \sum_{p \in H} (-x_p) g^p(y_1^o, y_2) \leq -\epsilon$

where a constraint of type 2) is added when an optimal solution is obtained for the sub-problem and a constraint of type 3) is added when the sub-problem is inconsistent.

Convergence to an ϵ-optimal solution in a finite number of steps using this decomposition technique is easily demonstrated. Assume we return from a sub-problem without achieving a bound ed gain in the incumbent solution when we obtained a feasible solution then

$$(11) \quad g^m(y_1^o, y_2) \geq g^m(y_1^c, y_2^c)$$

and

$$(12) \quad \sum_i (-x_i) \cdot g^i(y_1^o, y_2) > -\epsilon$$

if this set of dual variables is a repeat we contradict

$$(13) \quad g^m(y_1^o, y_2) + \sum_i (-x_i) g^i(y_1^o, y_2) \leq g^m(y_1^c, y_2^c) - \epsilon$$

which has been established from the master problem. Since there are only a finite number of dual solutions between bounded gains in a finite number of steps we exceed any lower bound or fail to solve the master. When we fail to solve the master we have established from one of the lower bound functions that we are within ϵ of the optimum.

When we return from the sub-problems with an inconsistent solution we have

$$(14) \quad \sum_{p \in H} (-x_p) \cdot g^p(y_1^o, y_2) > -g^w(y_1^o, y_2) - \epsilon$$

Again if w and H are a repeat we contradict

$$(15) \quad g^w(y_1^o, y_2) + \sum_{p \in H} (-x_p) g^p(y_1^o, y_2) \leq -\epsilon$$

which has been established from the master. So in at most a finite number of steps we resolve the inconsistency or demonstrate that the complete problem is inconsistent.

This type of decomposition has recently been employed in a large warehouse location problem (see [3]) and proved quite successful. It would seem to offer great promise in the area of regional water quality control.

IV. TRANSFER FUNCTIONS

Part 1: Theoretical Considerations

We begin by considering the two fundamental sets of mass balance differential equations. The most general form of these equations is given by Thomann [11] or as equations (1) and (2) by Pence, Jeglic and Thomann [8, pp. 383-83]. These basic D.E.'s characterize a one-dimensional estuary divided into n sections. The first set gives the transformation of polluter BOD to in-stream BOD and can be written as

$$V_i \frac{dL_i}{dt} = Q_{i-1,i}[\xi_{i-1,i}L_{i-1}+(1-\xi_{i-1,i})L_i]+E_{i-1,i}(L_{i-1}-L_i)$$

$$- Q_{i,i+1}[\xi_{i,i+1}L_i+(1-\xi_{i,i+1})L_{i+1}]+E_{i,i+1}(L_{i+1}-L_i)$$

$$- V_i dL_i+K_i \qquad i=1,2,\ldots,n$$

where the sections are numbered from upstream to downstream. L_i is the ultimate carbonaceous BOD in section i. $Q_{i-1,i}$ is the net flow from section i-1 to i. V_i is the volume of section i. $\xi_{i-1,i}$ is the advection coefficient from section i-1 to i. $E_{i-1,i}$ is the eddy exchange coefficient from section i-1 to i. d is the decay rate of BOD in all sections. K_i are all direct sources of BOD discharged to section i.

Since we require $dL_i/dt = 0$ in the steady state (SS) the above system can be written as follows

$$0 = r_{i,i-1}L_{i-1}+r_{ii}L_i+r_{i,i+1}L_{i+1}+K_i \qquad i=1,2,\ldots,n$$

where

$$r_{i,i-1} = \xi_{i-1,i}Q_{i-1,i}+E_{i-1,i}$$

$$r_{ii} = Q_{i-1,i}(1-\xi_{i-1,i})-Q_{i,i+1}\xi_{i,i+1}-E_{i-1,i}-E_{i,i+1}-V_i d$$

$$r_{i,i+1} = -(1-\xi_{i,i+1})Q_{i,i+1}+E_{i,i+1}$$

$$\text{and } r_{ij} = 0 \quad \text{if } |i-j| \geq 2$$

In matrix notation we have

$$RL + K = \odot$$

where R is a square matrix of order n. Assuming R is non-singular we have

$$L = -R^{-1}K$$

The second set of D.E.'s gives the transformation of stream

BOD into D.O. and can be written as

$$V_i \frac{dC_i}{dt} = Q_{i-1,i}[\xi_{i-1,i}C_{i-1}+(1-\xi_{i-1,i})C_i]+E_{i-1,i}(C_{i-1}-C_i)$$

$$- Q_{i,i+1}[(1-\xi_{i,i+1})C_{i+1}+\xi_{i,i+1}C_i]+E_{i,i+1}(C_{i+1}-C_i)$$

$$- V_i r_i C_i + V_i(r_i C_{S_i} - dL_i) \pm P_i \qquad i=1,\ldots,n$$

where C_i is the D.O. concentration in section i. r_i is the reaeration rate of D.O. in section i. C_{S_i} is the saturation concentration of D.O. in section i. P_i is all other sources and sinks of D.O. in section i. Since we require $dC_i/dt = 0$ in the steady state the above system can be written as follows

$$0 = s_{i,i-1}C_{i-1}+s_{ii}C_i+s_{i,i+1}C_{i+1}+V_i(r_i C_{s_i}-dL_i)\pm P_i \qquad i=1,\ldots,n$$

where

$$s_{i,i-1} = \xi_{i-1,i}Q_{i-1,i}+E_{i-1,i}$$

$$s_{ii} = Q_{i-1,i}(1-\xi_{i-1,i})-Q_{i,i+1}\xi_{i,i+1}-E_{i-1,i}-E_{i,i+1}-V_i r_i$$

$$s_{i,i+1} = -(1-\xi_{i,i+1})Q_{i,i+1}+E_{i,i+1}$$

$$\text{and } s_{ij} = 0 \quad \text{if } |i-j| \geq 2$$

Let V be a diagonal matrix of order n with the V_i as the diagonal elements

$$V = \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & & \vdots \\ \vdots & & & \\ 0 & & \cdots & V_n \end{bmatrix}$$

and let W be the column vector

$$W = \begin{bmatrix} r_1 C_{S_1} \\ r_2 C_{S_2} \\ \vdots \\ r_n C_{S_n} \end{bmatrix}$$

In matrix notation, the above system is

$$SC = dVL - V\,W \mp P$$

where S is a square matrix of order n. Assuming S is non-singular we have

$$C = -dS^{-1}V\,R^{-1}K - S^{-1}V\,W \mp S^{-1}P$$

The essential idea in calculating the SS transfer matrix is to inject 1 unit of BOD into each section and calculate the corresponding D.O. changes in each section. A symmetric interpretation is to remove 1 unit of BOD from each section. This gives a sign reversal in the final result. Replacing K above, by e_j (unit vector) and noting that the 2nd and 3rd terms on the right can be dropped we define

$$\phi_{\cdot j} = -dS^{-1}V\,R^{-1}e_j$$

where $\phi_{\cdot j}$ is the jth column of the transfer matrix. The transfer matrix Φ is then

$$\Phi = -dS^{-1}V\,R^{-1}I = -dS^{-1}V\,R^{-1} = -d(R\,V^{-1}S)^{-1}$$

The transfer matrix corresponding to a 1 unit removal in each section is

$$\Phi = d(R\,V^{-1}S)^{-1}$$

The mass balance D.E.'s hold for non-tidal streams as well as estuaries. It is an interesting exercise to show that the classical Streeter-Phelps equation is contained in these D.E.'s. The Streeter-Phelps equation can be derived as follows:

In the absence of reaeration assume that the rate of change of BOD at time t is proportional to the BOD at time t

$$\frac{dy_t}{dt} = -k_1 y_t \quad \text{or} \quad y_t' + k_1 y_t = 0$$

where y_t is the demand for oxygen (BOD) at time t and L is the ultimate BOD at time zero. The general solution to this D.E. is [1, Thm.1, p. 40]

$$\phi(t) = c\,e^{-k_1 t}$$

since $L = \phi(0) = c$, we have

$$y_t = \phi(t) = Le^{-k_1 t}$$

Similarly in the absence of demand for oxygen assume that the rate of change of D.O. deficit (DOD) is proportional to the

DOD at time t.

$$\frac{dD_t}{dt} = -k_2 D_t$$

where D_t is the D.O. deficit at time t, i.e., $D_t = C_S - C_t$. as before

$$D_t = D_0 e^{-k_2 t}$$

Combining the above D.E.'s we have the Streeter-Phelps D.E.

$$\frac{dD_t}{dt} = k_1 y_t - k_2 D_t \quad \text{or} \quad D_t' + k_2 D_t = k_1 y_t$$

The general solution is [1, Thm. 2, p. 41]

$$\phi(t) = e^{-k_2 t} \int_0^t e^{k_2 x} k_1 y_x dx + ce^{-k_2 t}$$

using $y_x = Le^{-k_1 x}$ we have

$$\phi(t) = \frac{k_1 L e^{-k_2 t}}{k_2 - k_1} \int_0^t e^{(k_2 - k_1)x} (k_2 - k_1) dx + ce^{-k_2 t}$$

$$= \frac{k_1 L}{k_2 - k_1} [e^{-k_1 t} - e^{-k_2 t}] + ce^{-k_2 t}$$

from the initial value problem we have $D_0 = \phi(0) = c$ and the classic Streeter-Phelps equation is

$$\phi(t) = D_t = \frac{k_1 L}{k_2 - k_1} [e^{-k_1 t} - e^{-k_2 t}] + D_0 e^{-k_2 t}$$

In the Streeter-Phelps D.E.Q. let $D_t = C_S - C_t$ giving

$$\frac{d(C_S - C_t)}{dt} = k_1 y_t - k_2 (C_S - C_t)$$

or

$$\frac{dC_t}{dt} = k_2 C_S - k_2 C_t - k_1 y_t \tag{16}$$

In the mass balance D.E. giving the transformation of stream

BOD to D.O. let $E_{ij} = 0$ and $\xi_{i-1,i} = 1$ which gives

$$(17) \quad \frac{dC_i}{dt} = \frac{Q_{i-1,i}C_{i-1}}{V_i} - \frac{Q_{i,i+1}C_i}{V_i} + r_iC_{S_i} - r_iC_i - dL_i \qquad (2)$$

Note that in (16) and (17) the following quantities are equivalent: $k_2 \sim r$, $k_1 \sim d$, and $y_t \sim L$. Comparing (16) and (17) we can see that the terms of (16) are contained in (17).

In the D.E. giving the transformation of stream BOD into D.O. we introducted the term P_i representing all other sources and sinks of D.O. in section i. Now let P_i represent sources of D.O. and Q_i sinks of D.O. in section i. From previous results we have

$$C = \Phi K - S^{-1}V\,W - S^{-1}P + S^{-1}Q$$

where P and Q are column vectors. The same basic idea used to construct a SS transfer matrix for BOD can be used to construct a SS transfer matrix for D.O. sources. We inject 1 unit of D.O. into each section and calculate the corresponding D.O. changes in each section. Define

$$\psi \cdot j = -S^{-1}e_j$$

where $\psi \cdot j$ is the jth column of the transfer matrix. The SS transfer matrix Ψ is then

$$\Psi = -S^{-1}I = -S^{-1}$$

This SS transfer matrix can be used to account for D.O. changes due to the following:

1) D.O. in the polluter discharges
2) D.O. in the regional plant discharges
3) in-stream aeration
4) flow augmentation.

For 1) and 2) let α_1 and β_1 be the average D.O. concentration of effluent from polluter i and regional plant i respectively. Thus 1) and 2) are handled by the following modification to the quality constraints

$$AFJ + A\sum_{i=1}^{k} \frac{t_{\cdot i}(1-r_i)p_i \cdot J}{1^T t_{\cdot i}} + C - A\overline{FJ} - A\sum_{i=1}^{k} \frac{\overline{t}_{\cdot i}(1-\overline{r}_i)\overline{p}_i \cdot \overline{J}}{1^T \overline{t}_{\cdot i}}$$

$$+ \Psi F \overline{\alpha} + \Psi T \overline{\beta} - \Psi \overline{F}\,\overline{\alpha} - \Psi \overline{T}\,\overline{\beta} \leq 0$$

In-stream aeration 3) requires the introduction of a variable specifying the amount of D.O. added in each section due to aeration. Flow augmentation requires a conservation of flow constraint for each reservoir and additional flow variables.

Part 2: Numerical Calculation of a Transfer Matrix

A general nonlinear programming problem for water quality control in a tidal estuary has been given by Graves, Hatfield and Whinston [4]. The flow chart [4, page 4] of the type of operations involved in the overall approach is given below:

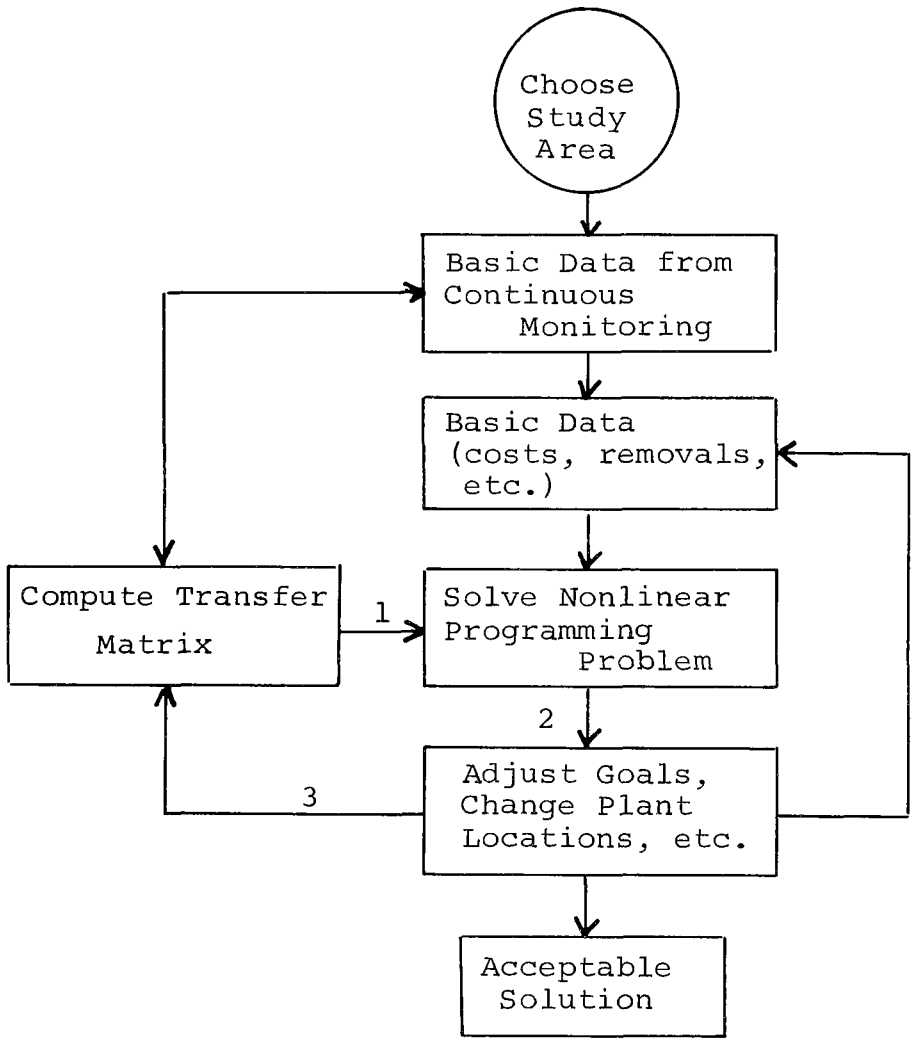


FIGURE 2

OPTIMIZATION PROCESS, ASSUMING FIXED TRANSFER MATRIX

In [4] the SS (Steady State) transfer coefficients were taken as a matrix of fixed constants and the path labeled 1 → 2 → 3 of Figure 2 was not investigated. There was no need to recompute the SS transfer coefficients since they were assumed constant. Unfortuantely the SS transfer matrix A, is not completely independent of the variables of the nonlinear programming problem. The calculation of the SS transfer matrix depends, among other things, on the estuarial intersection flows. Activating by-pass pipes and switching plant discharge points can change the estuarial flow pattern. This suggests some type of iterative process of: 1) solving the nonlinear programming problem; 2) using the current solution to update the transfer matrix; and 3) re-solving the nonlinear programming problem using the updated transfer matrix. With these ideas, we can modify the path 1 → 2 → 3 of Figure 2 as in Figure 3.

The purpose of this section is to develop the details of the calculations implied by Figure 3. We begin by considering the calculation of the transfer matrix. Let $Q_{i-1,i}$ be the net flow from section i-1 to i. V_i is the volume of section i. $\xi_{i-1,i}$ is the advection coefficient from section i-1 to i. $E_{i-1,i}$ is the eddy exchange coefficient from section i-1 to i. d is the decay rate of BOD in all sections. r_i is the reaeration rate of D.O. in section i. For computational purposes, a convenient system of units is $Q_{i-1,i}$ and $E_{i-1,i}$ in km^3/day, V_i in km^3 and d and r_i in 1/day. Note that $\xi_{i-1,i}$ is dimensionless.

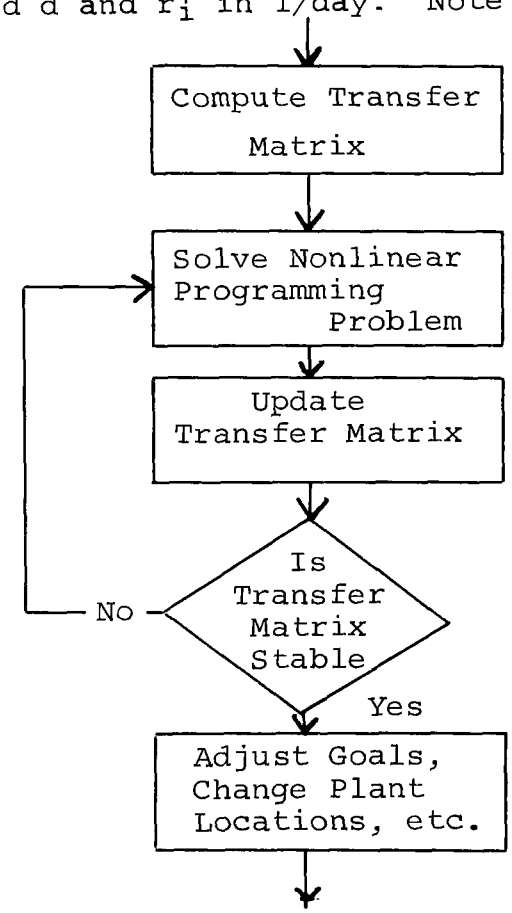


FIGURE 3

ITERATIVE PROCEDURE FOR FINDING STABLE TRANSFER MATRIX

Let R and S be square matrices of order n (n = the number of sections). If $r_{ij} \epsilon R$ and $s_{ij} \epsilon S$ and

$$(18) \quad c_{ii} = Q_{i-1,i}\,(1 - \xi_{i-1,i}) - Q_{i,i+1}\,\xi_{i,i+1} - E_{i-1,i} - E_{i,i+1}$$

then

$$r_{i,i-1} = s_{i,i-1} = \xi_{i-1,i}\,Q_{i-1,i} + E_{i-1,i}$$

$$r_{ii} = c_{ii} - V_i d$$

$$(19) \quad s_{ii} = c_{ii} - V_i\,r_i$$

$$r_{i,i+1} = s_{i,i+1} = -\,(1 - \xi_{i,i+1})\,Q_{i,i+1} + E_{i,i+1}$$

and $\quad r_{ij} = s_{ij} = 0 \quad \text{if } |i - j| \geq 2$

Let V be a diagonal matrix of order n with the V_i as the diagonal elements. Then the SS transfer matrix corresponding to a 1 lb/day BOD input into each section is given by

$$(20) \quad A = -(4.536 \cdot 10^{-7})\,d\,(R\,V^{-1}S)^{-1}$$

With the system of units chosen as above the elements of $R\,V^{-1}S$ will be close to one and accuracy will be maintained in calculating the inverse. The units of A are in mg/L/lb/day.

In order to calculate a SS transfer matrix for the Delaware Estuary, we need numerical values for the $Q_{i-1,i}$, $\xi_{i-1,i}$, $E_{i-1,i}$, V_i, r_i and d. These are in turn functions of the physical characteristics of the estuary. The necessary data on the physical properties of the estuary is given by Pence, Jeglic, and Thomann [8, page 390]. The values of $Q_{i-1,i} = Q(i)$ in km^3/day for a base flow of 3000 cfs are given in Table 1. Values of $E_{i-1,i}$ in km^3/day are calculated from (see [8] equation 6)

$$(21) \quad E_{i,i+1} = 1.57873\,\frac{K_{i,i+1}\,A_{i,i+1}}{\ell_i + \ell_{i+1}}$$

where $K_{i,i+1}$ is the eddy-diffusion coefficient from section i to i+1. $A_{i,i+1}$ is the cross-sectional area between section i and i+1; and ℓ_i is the length of section i. Values of $K_{i,i+1}$, $A_{i,i+1}$ and ℓ_i are given in [8, page 390]. The $E_{i-1,i} = E(i)$ are tabulated in Table 2. The advection coefficient is taken to be $\xi_{i-1,i} = \xi(i) = 0.5 = i=5,6,\ldots,31$. In sections one to four, the advective forces are stronger than the tidal action

	$Q(i) = Q_{i-1,i}$		
1.	7.34055E-03	17.	1.04040E-02
2.	7.43108E-03	18.	1.07123E-02
3.	7.67577E-03	19.	1.08591E-02
4.	7.69779E-03	20.	1.09081E-02
5.	7.77119E-03	21.	1.09081E-02
6.	7.86907E-03	22.	1.13778E-02
7.	8.35844E-03	23.	1.13803E-02
8.	7.69779E-03	24.	1.13925E-02
9.	7.74672E-03	25.	1.13925E-02
10.	7.77119E-03	26.	1.14047E-02
11.	8.29727E-03	27.	1.14170E-02
12.	8.37067E-03	28.	1.14170E-02
13.	8.37067E-03	29.	1.15393E-02
14.	8.48323E-03	30.	1.16127E-02
15.	8.95057E-03	31.	1.16861E-02
16.	9.92932E-03		

TABLE 1

NET FLOWS INTO SECTION i

	$E(i) = E_{i-1,i}$		
1.	9.77309E-04	17.	2.07010E-02
2.	2.43355E-03	18.	2.23784E-02
3.	3.37848E-03	19.	2.49241E-02
4.	3.88367E-03	20.	2.81803E-02
5.	4.49938E-03	21.	3.95734E-02
6.	5.38346E-03	22.	5.99522E-02
7.	8.71458E-03	23.	6.42937E-02
8.	1.56610E-02	24.	6.97798E-02
9.	2.02077E-02	25.	1.00399E-01
10.	2.18654E-02	26.	1.04101E-01
11.	2.40361E-02	27.	1.08521E-01
12.	2.56543E-02	28.	1.18523E-01
13.	2.60490E-02	29.	8.65670E-02
14.	2.86934E-02	30.	7.03126E-02
15.	2.32336E-02	31.	8.49277E-02
16.	1.93394E-02		

TABLE 2

EDDY EXCHANGE COEFFICIENTS BETWEEN SECTIONS i-1 AND i

and we take $\xi(1) = 0.86686$ $\xi(2) = 0.76969$ $\xi(3) = 0.61619$, and $\xi(4) = 0.52767$. Values of V_i in km^3 are taken from [8] pate 390 and are recorded in Table 3.

The reaeration rate r_i and the decay rate d are functions of water temperature. This has the effect of introducing a degree of freedom in the calculation of the SS transfer matrix since each matrix will depend on the particular value of water temperature chosen. Values of r_i in 1/day are calculated from (see [8] equations 8 and 9)

$$(22) \qquad r_i = \frac{24\ (8.1 \cdot 10^{-5}\ u_i)^{0.5}}{h_i^{1.5}}\ e^{.018\ (t\ -\ 20)}$$

where u_i is the mean tidal velocity in section i and h_i is the mean depth in section i. Values of u_i and h_i are given in [8] page 390. A value of $t=20^{\circ}$ $C=68^{\circ}$ F was chosen because: (a) it simplifies the calculation of (22); and (b) it is roughly the average water temperature from June to November. The r_i are tabulated in Table 4. Values of d in 1/day are calculated from (see [8] equation 10)

$$(23) \qquad d = 0.23 \cdot 1.099^{(t\ -20)}$$

With $t=20^{\circ}$ C we have d = 0.23/day.

The SS transfer matrix for the Delaware Estuary calculated from (20) is given in Table 5. The SS transfer matrix supplied to us by Prof. R.V. Thomann of Manhattan College, and used in our earlier work is given in Table 6 for comparison.

Updating the Transfer Matrix:

Recall that there are five sets of variables (the F, P, T, J and R matrices) in the nonlinear programming formulation. It is relatively simple calculation to update the intersectional flows $Q_{i,i+1}$, given the current F and T matrices and the original polluter discharges. If the base flow in the stream is large, changing the polluter discharge points effects, primarily, the intersectional flows. For the Delaware Estuary problem we are assuming that all other secondary effects are negligible. Some of the other quantities indirectly effected by changing the flow pattern are: 1) the advection coefficient $\xi_{i,i+1}$; and 2) the average depth h_i and consequently the reaeration rate r_i. This assumption is apparently justified for the Delaware Estuary; however, this is not always the case, especially in situations where polluter discharges make up a large percentage of the base flow. A detailed analysis of this particular aspect of the problem is given in Section V.

	V_i		
1.	6.85344E-03	17.	6.18509E-02
2.	1.03085E-02	18.	6.78547E-02
3.	1.30272E-02	19.	7.62374E-02
4.	1.50662E-02	20.	8.30342E-02
5.	1.80115E-02	21.	4.28198E-02
6.	2.14099E-02	22.	4.45757E-02
7.	1.28856E-02	23.	4.80874E-02
8.	1.42733E-02	24.	5.07494E-02
9.	1.50945E-02	25.	5.23920E-02
10.	1.64822E-02	26.	5.44877E-02
11.	1.78416E-02	27.	5.81693E-02
12.	1.85496E-02	28.	6.36634E-02
13.	1.96541E-02	29.	1.38655E-01
14.	2.27976E-02	30.	1.59158E-01
15.	5.26752E-02		
16.	5.74896E-02		

TABLE 3

VOLUMES OF SECTION i

	r_i		
1.	1.00116E-01	17.	1.30556E-01
2.	1.19462E-01	18.	1.12356E-01
3.	9.23170E-02	19.	1.12356E-01
4.	1.00116E-01	20.	1.12356E-01
5.	1.15604E-01	21.	1.21359E-01
6.	1.25953E-01	22.	9.93308E-02
7.	1.06598E-01	23.	1.21359E-01
8.	9.87044E-02	24.	1.21359E-01
9.	1.15604E-01	25.	1.41016E-01
10.	8.55555E-02	26.	1.95081E-01
11.	8.94839E-02	27.	1.78124E-01
12.	5.71866E-02	28.	1.78124E-01
13.	4.56965E-02	29.	1.29738E-01
14.	8.39498E-02	30.	1.20993E-01
15.	8.94839E-02		
16.	1.12356E-01		

TABLE 4

REAERATION RATE IN SECTION i

TABLE 5

TRANSFER MATRIX

	1	2	3	4	5	6	7	8	9	10
1	-1.018D-05	-1.588D-06	-1.113D-07	-3.784D-09	-2.616D-10	-3.347D-11	-1.035D-11	-5.375D-12	-3.149D-12	-1.872D-12
2	-1.778D-05	-1.244D-05	-1.096D-06	-4.082D-08	-2.955D-09	-3.884D-10	-1.216D-10	-6.367D-11	-3.752D-11	-2.241D-11
3	-2.380D-05	-2.090D-05	-1.437D-05	-6.919D-07	-5.545D-08	-7.686D-09	-2.465D-09	-1.309D-09	-7.791D-10	-4.693D-10
4	-2.594D-05	-2.489D-05	-2.206D-05	-1.506D-05	-1.578D-06	-2.435D-07	-8.153D-08	-4.435D-08	-2.684D-08	-1.640D-08
5	-2.401D-05	-2.412D-05	-2.369D-05	-2.127D-05	-1.542D-05	-3.053D-06	-1.107D-06	-6.267D-07	-3.896D-07	-2.430D-07
6	-1.934D-05	-1.998D-05	-2.075D-05	-2.087D-05	-1.951D-05	-1.520D-05	-6.618D-06	-4.047D-06	-2.635D-06	-1.701D-06
7	-1.721D-05	-1.802D-05	-1.919D-05	-2.020D-05	-2.047D-05	-1.911D-05	-1.496D-05	-1.065D-05	-7.484D-06	-5.086D-06
8	-1.494D-05	-1.577D-05	-1.704D-05	-1.838D-05	-1.938D-05	-1.949D-05	-1.770D-05	-1.536D-05	-1.167D-05	-8.306D-06
9	-1.302D-05	-1.381D-05	-1.509D-05	-1.655D-05	-1.790D-05	-1.880D-05	-1.838D-05	-1.724D-05	-1.516D-05	-1.160D-05
10	-1.136D-05	-1.211D-05	-1.333D-05	-1.482D-05	-1.635D-05	-1.773D-05	-1.819D-05	-1.785D-05	-1.686D-05	-1.492D-05
11	-1.004D-05	-1.075D-05	-1.191D-05	-1.338D-05	-1.499D-05	-1.663D-05	-1.763D-05	-1.778D-05	-1.750D-05	-1.661D-05
12	-8.906D-06	-9.561D-06	-1.065D-05	-1.206D-05	-1.366D-05	-1.541D-05	-1.672D-05	-1.718D-05	-1.735D-05	-1.714D-05
13	-7.633D-06	-8.215D-06	-9.194D-06	-1.048D-05	-1.198D-05	-1.369D-05	-1.511D-05	-1.574D-05	-1.618D-05	-1.642D-05
14	-6.267D-06	-6.759D-06	-7.591D-06	-8.701D-06	-1.002D-05	-1.157D-05	-1.294D-05	-1.362D-05	-1.419D-05	-1.467D-05
15	-4.581D-06	-4.954D-06	-5.591D-06	-6.454D-06	-7.500D-06	-8.776D-06	-9.976D-06	-1.063D-05	-1.124D-05	-1.188D-05
16	-3.033D-06	-3.289D-06	-3.726D-06	-4.328D-06	-5.071D-06	-5.999D-06	-6.911D-06	-7.435D-06	-7.961D-06	-8.544D-06
17	-1.945D-06	-2.113D-06	-2.402D-06	-2.804D-06	-3.306D-06	-3.944D-06	-4.590D-06	-4.973D-06	-5.371D-06	-5.830D-06
18	-1.262D-06	-1.373D-06	-1.566D-06	-1.835D-06	-2.174D-06	-2.612D-06	-3.063D-06	-3.338D-06	-3.629D-06	-3.972D-06
19	-8.037D-07	-8.758D-07	-1.001D-06	-1.176D-06	-1.400D-06	-1.690D-06	-1.995D-06	-2.183D-06	-2.385D-06	-2.627D-06
20	-5.071D-07	-5.531D-07	-6.330D-07	-7.461D-07	-8.905D-07	-1.080D-06	-1.280D-06	-1.405D-06	-1.541D-06	-1.706D-06
21	-3.597D-07	-3.925D-07	-4.497D-07	-5.307D-07	-6.344D-07	-7.709D-07	-9.163D-07	-1.008D-06	-1.107D-06	-1.228D-06
22	-2.768D-07	-3.022D-07	-3.464D-07	-4.092D-07	-4.898D-07	-5.960D-07	-7.096D-07	-7.811D-07	-8.595D-07	-9.551D-07
23	-2.060D-07	-2.249D-07	-2.580D-07	-3.050D-07	-3.655D-07	-4.453D-07	-5.310D-07	-5.851D-07	-6.445D-07	-7.173D-07
24	-1.486D-07	-1.623D-07	-1.863D-07	-2.204D-07	-2.643D-07	-3.225D-07	-3.850D-07	-4.247D-07	-4.683D-07	-5.219D-07
25	-1.131D-07	-1.236D-07	-1.419D-07	-1.679D-07	-2.016D-07	-2.462D-07	-2.942D-07	-3.247D-07	-3.584D-07	-3.998D-07
26	-8.392D-08	-9.173D-08	-1.054D-07	-1.248D-07	-1.499D-07	-1.832D-07	-2.192D-07	-2.420D-07	-2.673D-07	-2.985D-07
27	-6.213D-08	-6.793D-08	-7.805D-08	-9.249D-08	-1.112D-07	-1.360D-07	-1.628D-07	-1.799D-07	-1.989D-07	-2.222D-07
28	-4.621D-08	-5.053D-08	-5.807D-08	-6.885D-08	-8.279D-08	-1.014D-07	-1.215D-07	-1.343D-07	-1.485D-07	-1.661D-07
29	-2.877D-08	-3.147D-08	-3.619D-08	-4.293D-08	-5.167D-08	-6.332D-08	-7.598D-08	-8.408D-08	-9.307D-08	-1.042D-07
30	-1.245D-08	-1.362D-08	-1.567D-08	-1.860D-08	-2.240D-08	-2.747D-08	-3.299D-08	-3.652D-08	-4.046D-08	-4.533D-08

TABLE 5 Cont.

TRANSFER MATRIX

	11	12	13	14	15	16	17	18	19	20
1	-1.139D-12	-7.023D-13	-4.245D-13	-2.590D-13	-1.295D-13	-5.298D-14	-2.142D-14	-8.676D-15	-3.599D-15	-1.554D-15
2	-1.369D-11	-8.471D-12	-5.136D-12	-3.142D-12	-1.577D-12	-6.468D-13	-2.620D-13	-1.063D-13	-4.415D-14	-1.908D-14
3	-2.888D-10	-1.798D-10	-1.096D-10	-6.732D-11	-3.398D-11	-1.400D-11	-5.693D-12	-2.316D-12	-9.639D-13	-4.172D-13
4	-1.020D-08	-6.411D-09	-3.939D-09	-2.436D-09	-1.240D-09	-5.149D-10	-2.104D-10	-8.592D-11	-3.586D-11	-1.556D-11
5	-1.538D-07	-9.790D-08	-6.084D-08	-3.798D-08	-1.956D-08	-8.197D-09	-3.372D-09	-1.384D-09	-5.800D-10	-2.524D-10
6	-1.105D-06	-7.174D-07	-4.531D-07	-2.866D-07	-1.500D-07	-6.367D-08	-2.642D-08	-1.092D-08	-4.598D-09	-2.008D-09
7	-3.424D-06	-2.282D-06	-1.471D-06	-9.458D-07	-5.046D-07	-2.172D-07	-9.105D-08	-3.789D-08	-1.604D-08	-7.036D-09
8	-5.759D-06	-3.918D-06	-2.566D-06	-1.668D-06	-9.021D-07	-3.923D-07	-1.655D-07	-6.921D-08	-2.941D-08	-1.293D-08
9	-8.399D-06	-5.874D-06	-3.925D-06	-2.590D-06	-1.424D-06	-6.265D-07	-2.664D-07	-1.120D-07	-4.779D-08	-2.108D-08
10	-1.160D-05	-8.467D-06	-5.825D-06	-3.922D-06	-2.204D-06	-9.849D-07	-4.229D-07	-1.791D-07	-7.679D-08	-3.399D-08
11	-1.478D-05	-1.154D-05	-8.284D-06	-5.736D-06	-3.317D-06	-1.511D-06	-6.568D-07	-2.804D-07	-1.210D-07	-5.376D-08
12	-1.629D-05	-1.447D-05	-1.113D-05	-8.031D-06	-4.834D-06	-2.259D-06	-9.965D-07	-4.298D-07	-1.868D-07	-8.344D-08
13	-1.626D-05	-1.545D-05	-1.363D-05	-1.055D-05	-6.751D-06	-3.268D-06	-1.471D-06	-6.432D-07	-2.821D-07	-1.268D-07
14	-1.491D-05	-1.477D-05	-1.399D-05	-1.238D-05	-8.728D-06	-4.442D-06	-2.054D-06	-9.131D-07	-4.050D-07	-1.836D-07
15	-1.243D-05	-1.283D-05	-1.296D-05	-1.268D-05	-1.157D-05	-6.529D-06	-3.173D-06	-1.452D-06	-6.566D-07	-3.015D-07
16	-9.129D-06	-9.687D-06	-1.018D-05	-1.052D-05	-1.071D-05	-9.759D-06	-5.539D-06	-2.744D-06	-1.300D-06	-6.152D-07
17	-6.318D-06	-6.829D-06	-7.359D-06	-7.849D-06	-8.443D-06	-9.068D-06	-8.592D-06	-5.026D-06	-2.586D-06	-1.283D-06
18	-4.350D-06	-4.764D-06	-5.222D-06	-5.687D-06	-6.328D-06	-7.399D-06	-8.291D-06	-8.126D-06	-4.916D-06	-2.635D-06
19	-2.899D-06	-3.205D-06	-3.555D-06	-3.927D-06	-4.467D-06	-5.490D-06	-6.676D-06	-7.674D-06	-7.596D-06	-4.744D-06
20	-1.893D-06	-2.107D-06	-2.357D-06	-2.629D-06	-3.035D-06	-3.851D-06	-4.906D-06	-6.085D-06	-7.015D-06	-6.989D-06
21	-1.367D-06	-1.527D-06	-1.715D-06	-1.922D-06	-2.235D-06	-2.878D-06	-3.743D-06	-4.787D-06	-5.817D-06	-6.469D-06
22	-1.065D-06	-1.192D-06	-1.343D-06	-1.510D-06	-1.764D-06	-2.294D-06	-3.022D-06	-3.937D-06	-4.930D-06	-5.793D-06
23	-8.012D-07	-8.988D-07	-1.015D-06	-1.145D-06	-1.343D-06	-1.760D-06	-2.343D-06	-3.100D-06	-3.972D-06	-4.857D-06
24	-5.838D-07	-6.562D-07	-7.429D-07	-8.397D-07	-9.886D-07	-1.305D-06	-1.755D-06	-2.352D-06	-3.073D-06	-3.876D-06
25	-4.477D-07	-5.040D-07	-5.714D-07	-6.471D-07	-7.638D-07	-1.014D-06	-1.372D-06	-1.855D-06	-2.454D-06	-3.158D-06
26	-3.346D-07	-3.771D-07	-4.283D-07	-4.858D-07	-5.748D-07	-7.666D-07	-1.044D-06	-1.422D-06	-1.903D-06	-2.490D-06
27	-2.494D-07	-2.814D-07	-3.201D-07	-3.636D-07	-4.313D-07	-5.777D-07	-7.909D-07	-1.085D-06	-1.467D-06	-1.948D-06
28	-1.866D-07	-2.108D-07	-2.400D-07	-2.730D-07	-3.245D-07	-4.363D-07	-6.002D-07	-8.287D-07	-1.129D-06	-1.518D-06
29	-1.172D-07	-1.326D-07	-1.512D-07	-1.724D-07	-2.055D-07	-2.778D-07	-3.846D-07	-5.355D-07	-7.381D-07	-1.008D-06
30	-5.104D-08	-5.780D-08	-6.603D-08	-7.537D-08	-9.001D-08	-1.222D-07	-1.700D-07	-2.381D-07	-3.308D-07	-4.566D-07

TABLE 5 Cont.

TRANSFER MATRIX

	21	22	23	24	25	26	27	28	29	30
1	-8.449D-16	-5.398D-16	-3.430D-16	-2.162D-16	-1.522D-16	-1.062D-16	0.0	0.0	0.0	0.0
2	-1.038D-14	-6.635D-15	-4.218D-15	-2.659D-15	-1.872D-15	-1.306D-15	-9.005D-16	-6.213D-16	-3.388D-16	-1.234D-16
3	-2.272D-13	-1.453D-13	-9.240D-14	-5.828D-14	-4.105D-14	-2.865D-14	-1.975D-14	-1.363D-14	-7.434D-15	-2.709D-15
4	-8.482D-12	-5.428D-12	-3.455D-12	-2.181D-12	-1.537D-12	-1.073D-12	-7.399D-13	-5.108D-13	-2.787D-13	-1.016D-13
5	-1.378D-10	-8.827D-11	-5.624D-11	-3.552D-11	-2.504D-11	-1.749D-11	-1.207D-11	-8.335D-12	-4.550D-12	-1.659D-12
6	-1.098D-09	-7.046D-10	-4.494D-10	-2.842D-10	-2.005D-10	-1.401D-10	-9.675D-11	-6.684D-11	-3.651D-11	-1.332D-11
7	-3.857D-09	-2.477D-09	-1.582D-09	-1.001D-09	-7.071D-10	-4.946D-10	-3.416D-10	-2.361D-10	-1.291D-10	-4.712D-11
8	-7.097D-09	-4.563D-09	-2.916D-09	-1.847D-09	-1.305D-09	-9.131D-10	-6.310D-10	-4.363D-10	-2.386D-10	-8.712D-11
9	-1.159D-08	-7.457D-09	-4.770D-09	-3.024D-09	-2.137D-09	-1.496D-09	-1.034D-09	-7.154D-10	-3.914D-10	-1.430D-10
10	-1.872D-08	-1.206D-08	-7.724D-09	-4.901D-09	-3.467D-09	-2.428D-09	-1.679D-09	-1.162D-09	-6.361D-10	-2.325D-10
11	-2.967D-08	-1.915D-08	-1.228D-08	-7.800D-09	-5.521D-09	-3.869D-09	-2.678D-09	-1.854D-09	-1.015D-09	-3.713D-10
12	-4.617D-08	-2.985D-08	-1.917D-08	-1.219D-08	-8.637D-09	-6.058D-09	-4.195D-09	-2.906D-09	-1.593D-09	-5.829D-10
13	-7.041D-08	-4.562D-08	-2.934D-08	-1.870D-08	-1.326D-08	-9.309D-09	-6.452D-09	-4.472D-09	-2.454D-09	-8.986D-10
14	-1.023D-07	-6.644D-08	-4.284D-08	-2.734D-08	-1.942D-08	-1.365D-08	-9.468D-09	-6.568D-09	-3.608D-09	-1.322D-09
15	-1.690D-07	-1.102D-07	-7.131D-08	-4.565D-08	-3.249D-08	-2.287D-08	-1.589D-08	-1.104D-08	-6.075D-09	-2.229D-09
16	-3.498D-07	-2.302D-07	-1.500D-07	-9.669D-08	-6.910D-08	-4.883D-08	-3.404D-08	-2.371D-08	-1.310D-08	-4.819D-09
17	-7.447D-07	-4.965D-07	-3.271D-07	-2.128D-07	-1.530D-07	-1.087D-07	-7.607D-08	-5.317D-08	-2.952D-08	-1.090D-08
18	-1.578D-06	-1.072D-06	-7.172D-07	-4.723D-07	-3.423D-07	-2.448D-07	-1.724D-07	-1.210D-07	-6.763D-08	-2.510D-08
19	-2.994D-06	-2.095D-06	-1.434D-06	-9.616D-07	-7.050D-07	-5.089D-07	-3.612D-07	-2.552D-07	-1.438D-07	-5.371D-08
20	-4.922D-06	-3.637D-06	-2.588D-06	-1.788D-06	-1.335D-06	-9.772D-07	-7.017D-07	-5.005D-07	-2.856D-07	-1.076D-07
21	-6.005D-06	-4.930D-06	-3.745D-06	-2.708D-06	-2.075D-06	-1.550D-06	-1.131D-06	-8.165D-07	-4.733D-07	-1.803D-07
22	-5.974D-06	-5.635D-06	-4.600D-06	-3.478D-06	-2.728D-06	-2.073D-06	-1.533D-06	-1.118D-06	-6.566D-07	-2.523D-07
23	-5.354D-06	-5.430D-06	-5.099D-06	-4.151D-06	-3.375D-06	-2.630D-06	-1.981D-06	-1.465D-06	-8.745D-07	-3.398D-07
24	-4.482D-06	-4.760D-06	-4.823D-06	-4.543D-06	-3.925D-06	-3.180D-06	-2.463D-06	-1.856D-06	-1.134D-06	-4.471D-07
25	-3.757D-06	-4.094D-06	-4.308D-06	-4.321D-06	-4.083D-06	-3.483D-06	-2.788D-06	-2.149D-06	-1.347D-06	-5.393D-07
26	-3.031D-06	-3.369D-06	-3.646D-06	-3.814D-06	-3.802D-06	-3.584D-06	-3.041D-06	-2.429D-06	-1.581D-06	-6.476D-07
27	-2.418D-06	-2.731D-06	-3.020D-06	-3.257D-06	-3.365D-06	-3.367D-06	-3.184D-06	-2.698D-06	-1.860D-06	-7.859D-07
28	-1.913D-06	-2.187D-06	-2.458D-06	-2.709D-06	-2.867D-06	-2.975D-06	-2.984D-06	-2.811D-06	-2.116D-06	-9.233D-07
29	-1.295D-06	-1.503D-06	-1.722D-06	-1.947D-06	-2.115D-06	-2.280D-06	-2.418D-06	-2.480D-06	-2.372D-06	-1.148D-06
30	-5.945D-07	-6.970D-07	-8.083D-07	-9.277D-07	-1.024D-06	-1.127D-06	-1.230D-06	-1.312D-06	-1.373D-06	-1.103D-06

TABLE 6

THOMANN'S TRANSFER MATRIX

	1	2	3	4	5	6	7	8	9	10
1	7.834E-06	9.331E-07	5.767E-08	1.343E-09	3.271E-11	9.865E-13	2.057E-13	8.947E-14	3.855E-14	1.687E-14
2	1.387E-05	1.069E-05	8.356E-07	2.130E-08	5.439E-10	1.685E-11	3.557E-12	1.559E-12	6.758E-13	2.973E-13
3	1.771E-05	1.706E-05	1.240E-05	4.084E-07	1.157E-08	3.778E-10	8.166E-11	3.626E-11	1.589E-11	7.057E-12
4	1.806E-05	1.892E-05	1.770E-05	1.274E-05	4.770E-07	1.738E-08	3.921E-09	1.782E-09	7.953E-10	3.587E-10
5	1.616E-05	1.768E-05	1.834E-05	1.750E-05	1.330E-05	6.325E-07	1.548E-07	7.326E-08	3.369E-08	1.557E-08
6	1.159E-05	1.302E-05	1.428E-05	1.528E-05	1.529E-05	1.227E-05	3.729E-06	1.924E-06	9.372E-07	4.524E-07
7	9.761E-06	1.110E-05	1.248E-05	1.399E-05	1.523E-05	1.519E-05	1.199E-05	7.489E-06	4.044E-06	2.089E-06
8	8.196E-06	9.389E-06	1.070E-05	1.229E-05	1.393E-05	1.510E-05	1.435E-05	1.221E-05	7.409E-06	4.081E-06
9	6.633E-06	7.643E-06	8.808E-06	1.030E-06	1.203E-05	1.378E-06	1.446E-05	1.383E-05	1.177E-05	7.418E-06
10	5.383E-06	6.234E-06	7.251E-06	8.612E-06	1.029E-05	1.227E-05	1.371E-05	1.396E-05	1.356E-05	1.191E-05
11	4.571E-06	5.311E-06	6.217E-06	7.459E-06	9.047E-06	1.105E-05	1.280E-05	1.346E-05	1.386E-05	1.356E-06
12	3.991E-06	4.650E-06	5.469E-06	6.611E-06	8.108E-06	1.008E-06	1.196E-05	1.283E-05	1.369E-06	1.416E-05
13	3.384E-06	3.951E-06	4.664E-06	5.672E-06	7.015E-06	8.833E-06	1.066E-05	1.161E-06	1.267E-05	1.356E-05
14	2.779E-06	3.250E-06	3.848E-06	4.701E-06	5.850E-06	7.437E-06	9.091E-06	9.994E-06	1.108E-05	1.213E-05
15	2.016E-06	2.362E-06	2.806E-06	3.447E-06	4.321E-06	5.555E-06	6.887E-06	7.655E-06	8.630E-06	9.672E-06
16	1.382E-06	1.621E-06	1.931E-06	2.381E-06	3.002E-06	3.891E-06	4.873E-06	5.458E-06	6.224E-06	7.083E-06
17	9.352E-07	1.098E-06	1.311E-06	1.621E-06	2.051E-06	2.672E-06	3.368E-06	3.790E-06	4.353E-06	5.000E-06
18	6.203E-07	7.291E-07	8.710E-07	1.079E-06	1.368E-06	1.789E-06	2.265E-06	2.557E-06	2.950E-06	3.408E-06
19	4.028E-07	4.736E-07	5.663E-07	7.023E-07	8.921E-07	1.169E-06	1.484E-06	1.679E-06	1.943E-06	2.253E-06
20	2.577E-07	3.031E-07	3.625E-07	4.499E-07	5.722E-07	7.509E-07	9.549E-07	1.082E-06	1.254E-06	1.458E-06
21	1.835E-07	2.159E-07	2.583E-07	3.207E-07	4.081E-07	5.359E-07	6.820E-07	7.731E-07	8.970E-07	1.044E-06
22	1.385E-07	1.629E-07	1.950E-07	2.422E-07	3.082E-07	4.049E-07	5.156E-07	5.846E-07	6.788E-07	7.905E-07
23	9.923E-08	1.167E-07	1.397E-07	1.736E-07	2.209E-07	2.904E-07	3.699E-07	4.196E-07	4.873E-07	5.679E-07
24	6.771E-08	7.967E-08	9.537E-08	1.185E-07	1.508E-07	1.983E-07	2.527E-07	2.867E-07	3.332E-07	3.885E-07
25	4.750E-08	5.589E-08	6.691E-08	8.313E-08	1.059E-07	1.392E-07	1.774E-07	2.014E-07	2.341E-07	2.730E-07
26	3.218E-08	3.786E-08	4.533E-08	5.633E-08	7.174E-08	9.436E-08	1.203E-07	1.366E-07	1.588E-07	1.852E-07
27	2.187E-08	2.574E-08	3.081E-08	3.829E-08	4.878E-08	6.417E-08	8.184E-08	9.291E-08	1.080E-07	1.261E-07
28	1.512E-08	1.780E-08	2.131E-08	2.648E-08	3.374E-08	4.439E-08	5.663E-08	6.430E-08	7.479E-08	8.732E-08
29	8.658E-09	1.019E-08	1.220E-08	1.517E-08	1.933E-08	2.544E-08	3.246E-08	3.687E-08	4.290E-08	5.011E-08
30	3.766E-09	4.432E-09	5.308E-09	6.599E-09	8.410E-09	1.107E-08	1.413E-08	1.606E-08	1.869E-08	2.184E-08

TABLE 6 Cont.

THOMANN'S TRANSFER MATRIX

	11	12	13	14	15	16	17	18	19	20
1	8.916E-15	4.846E-15	2.598E-15	1.423E-15	6.122E-16	2.085E-16	7.202E-17	2.539E-17	9.338E-18	3.625E-18
2	1.578E-13	8.605E-14	4.627E-14	2.540E-14	1.096E-14	3.744E-15	1.295E-15	4.573E-16	1.683E-16	6.537E-17
3	3.771E-12	2.069E-12	1.118E-12	6.161E-13	2.674E-13	9.170E-14	3.182E-14	1.125E-14	4.147E-15	1.612E-15
4	1.938E-10	1.073E-10	5.843E-11	3.241E-11	1.418E-11	4.894E-12	1.705E-12	6.049E-13	2.233E-13	8.693E-14
5	8.555E-09	4.800E-09	2.644E-09	1.480E-09	6.550E-10	2.280E-10	7.992E-11	2.845E-11	1.053E-11	4.107E-12
6	2.557E-07	1.466E-07	8.220E-08	4.666E-08	2.101E-08	7.406E-09	2.617E-09	9.368E-10	3.480E-10	1.360E-10
7	1.229E-06	7.255E-07	4.161E-07	2.403E-07	1.104E-07	3.950E-08	1.409E-08	5.073E-09	1.892E-09	7.413E-10
8	2.486E-06	1.502E-06	8.767E-07	5.128E-07	2.390E-07	8.640E-08	3.101E-08	1.121E-08	4.193E-09	1.646E-09
9	4.810E-06	3.023E-06	1.814E-06	1.082E-06	5.152E-07	1.890E-07	6.842E-08	2.488E-08	9.337E-09	3.673E-09
10	8.634E-06	5.767E-06	3.600E-06	2.205E-06	1.080E-06	4.032E-07	1.476E-07	5.402E-08	2.036E-08	8.031E-09
11	1.213E-06	8.872E-06	5.834E-06	3.690E-06	1.865E-06	7.105E-07	2.630E-07	9.695E-08	3.670E-08	1.452E-08
12	1.382E-05	1.235E-05	8.909E-06	5.930E-06	3.141E-06	1.229E-06	4.620E-07	1.719E-07	6.543E-08	2.597E-08
13	1.389E-05	1.356E-05	1.204E-05	8.782E-06	5.006E-06	2.037E-06	7.815E-07	2.942E-07	1.128E-07	4.500E-08
14	1.278E-05	1.311E-05	1.274E-05	1.130E-05	7.287E-06	3.139E-06	1.238E-06	4.733E-07	1.832E-07	7.349E-08
15	1.049E-05	1.124E-05	1.175E-05	1.180E-05	1.096E-05	5.328E-06	2.214E-06	8.700E-07	3.422E-07	1.386E-07
16	7.821E-06	8.603E-06	9.361E-06	9.965E-06	1.047E-05	9.817E-06	4.854E-06	2.063E-06	8.459E-07	3.508E-07
17	5.580E-06	6.231E-06	6.928E-06	7.593E-06	8.421E-06	9.431E-06	9.108E-06	4.615E-06	2.047E-06	8.839E-07
18	3.829E-06	4.315E-06	4.859E-06	5.415E-06	6.182E-06	7.500E-06	8.643E-06	8.504E-06	4.484E-06	2.088E-06
19	2.542E-06	2.881E-06	3.271E-06	3.681E-06	4.274E-06	5.413E-06	6.747E-06	7.897E-06	7.892E-06	4.340E-06
20	1.649E-06	1.877E-06	2.141E-06	2.424E-06	2.844E-06	3.691E-06	4.794E-06	6.051E-06	7.143E-06	7.221E-06
21	1.182E-06	1.347E-06	1.540E-06	1.749E-06	2.061E-06	2.703E-06	3.568E-06	4.630E-06	5.760E-06	6.569E-06
22	8.960E-07	1.022E-06	1.170E-06	1.331E-06	1.572E-06	2.075E-06	2.766E-06	3.648E-06	4.669E-06	5.640E-06
23	6.440E-07	7.352E-07	8.425E-07	9.597E-07	1.136E-06	1.507E-06	2.025E-06	2.705E-06	3.536E-06	4.446E-06
24	4.408E-07	5.036E-07	5.776E-07	6.588E-07	7.814E-07	1.041E-06	1.408E-06	1.900E-06	2.528E-06	3.278E-06
25	3.099E-07	3.542E-07	4.066E-07	4.641E-07	5.512E-07	7.368E-07	1.0[illegible]1E-06	1.361E-06	1.833E-06	2.425E-06
26	2.103E-07	2.406E-07	2.763E-07	3.157E-07	3.754E-07	5.033E-07	6.871E-07	9.407E-07	1.280E-06	1.725E-06
27	1.432E-07	1.639E-07	1.884E-07	2.154E-07	2.565E-07	3.449E-07	4.729E-07	6.517E-07	8.959E-07	1.227E-06
28	9.923E-08	1.136E-07	1.306E-07	1.495E-07	1.782E-07	2.403E-07	3.307E-07	4.584E-07	6.359E-07	8.834E-07
29	5.697E-08	6.526E-08	7.512E-08	8.606E-08	1.028E-07	1.391E-07	1.926E-07	2.693E-07	3.783E-07	5.360E-07
30	2.484E-08	2.847E-08	3.279E-08	3.760E-08	4.497E-08	6.106E-08	8.491E-08	1.195E-07	1.695E-07	2.437E-07

TABLE 6 Cont.

THOMANN'S TRANSFER MATRIX

	21	22	23	24	25	26	27	28	29	30
1	1.829E-18	1.104E-18	6.589E-19	3.874E-19	2.542E-19	1.644E-19	1.048E-19	6.734E-20	3.316E-20	1.177E-20
2	3.298E-17	1.991E-17	1.189E-17	6.989E-18	4.586E-18	2.966E-18	1.892E-18	1.215E-18	5.983E-19	2.124E-19
3	8.137E-16	4.913E-16	2.934E-16	1.725E-16	1.132E-16	7.323E-17	4.671E-17	3.001E-17	1.478E-17	5.246E-18
4	4.390E-14	2.651E-14	1.584E-14	9.316E-15	6.114E-15	3.955E-15	2.523E-15	1.621E-15	7.983E-16	2.835E-16
5	2.075E-12	1.254E-12	7.495E-13	4.409E-13	2.894E-13	1.873E-13	1.195E-13	7.676E-14	3.781E-14	1.343E-14
6	6.881E-11	4.160E-11	2.487E-11	1.464E-11	9.613E-12	6.220E-12	3.969E-12	2.551E-12	1.257E-12	4.463E-13
7	3.755E-10	2.272E-10	1.359E-10	8.002E-11	5.256E-11	3.402E-11	2.171E-11	1.395E-11	6.875E-12	2.442E-12
8	8.340E-10	5.048E-10	3.021E-10	1.780E-10	1.169E-10	7.567E-11	4.830E-11	3.105E-11	1.530E-11	5.436E-12
9	1.864E-09	1.129E-09	6.759E-10	3.982E-10	2.617E-10	1.694E-10	1.082E-10	6.953E-11	3.427E-11	1.218E-11
10	4.079E-09	2.473E-09	1.481E-09	8.732E-10	5.739E-10	3.717E-10	2.373E-10	1.526E-10	7.523E-11	2.674E-11
11	7.384E-09	4.479E-09	2.685E-09	1.583E-09	1.041E-09	6.743E-10	4.307E-10	2.769E-10	1.336E-10	4.854E-11
12	1.323E-08	8.032E-09	4.818E-09	2.843E-09	1.870E-09	1.212E-09	7.741E-10	4.979E-10	2.456E-10	8.731E-11
13	2.297E-08	1.396E-08	8.382E-09	4.950E-09	3.257E-09	2.111E-09	1.349E-09	8.680E-10	4.283E-10	1.523E-10
14	3.760E-08	2.289E-08	1.376E-08	8.133E-09	5.354E-09	3.472E-09	2.220E-09	1.429E-09	7.052E-10	2.509E-10
15	7.118E-08	4.344E-08	2.616E-08	1.549E-08	1.021E-08	6.623E-09	4.237E-09	2.728E-09	1.348E-09	4.797E-10
16	1.820E-07	1.117E-07	6.758E-08	4.015E-08	2.652E-08	1.724E-08	1.104E-08	7.120E-09	3.523E-09	1.256E-09
17	4.660E-07	2.888E-07	1.759E-07	1.051E-07	6.966E-08	4.541E-08	2.916E-08	1.883E-08	9.342E-09	3.336E-09
18	1.132E-06	7.123E-07	4.390E-07	2.646E-07	1.763E-07	1.154E-07	7.438E-08	4.817E-08	2.399E-08	8.593E-09
19	2.477E-06	1.602E-06	1.007E-06	6.159E-07	4.141E-07	2.730E-07	1.769E-07	1.151E-07	5.767E-08	2.075E-08
20	4.645E-06	3.178E-06	2.074E-06	1.302E-06	8.890E-07	5.931E-07	3.880E-07	2.543E-07	1.287E-07	4.665E-08
21	6.074E-06	4.699E-06	3.290E-06	2.162E-06	1.515E-06	1.029E-06	6.833E-07	4.528E-07	2.326E-07	8.524E-08
22	5.908E-06	5.514E-06	4.204E-06	2.901E-06	2.084E-06	1.442E-06	9.701E-07	6.494E-07	3.379E-07	1.250E-07
23	5.020E-06	5.119E-06	4.747E-06	3.587E-06	2.689E-06	1.913E-06	1.314E-06	8.924E-07	4.731E-07	1.772E-07
24	3.899E-06	4.196E-06	4.283E-06	3.998E-06	3.247E-06	2.425E-06	1.720E-06	1.195E-06	6.513E-07	2.485E-07
25	2.979E-06	3.306E-06	3.543E-06	3.606E-06	3.387E-06	2.722E-06	2.020E-06	1.445E-06	8.143E-07	3.173E-07
26	2.177E-06	2.475E-06	2.750E-06	2.962E-06	3.014E-06	2.866E-06	2.318E-06	1.744E-06	1.036E-06	4.165E-07
27	1.586E-06	1.841E-06	2.105E-06	2.363E-06	2.533E-06	2.636E-06	2.563E-06	2.106E-06	1.356E-06	5.691E-07
28	1.164E-06	1.373E-06	1.605E-06	1.856E-06	2.059E-06	2.259E-06	2.399E-06	2.352E-06	1.720E-06	7.655E-07
29	7.254E007	8.735E-07	1.048E-06	1.255E-06	1.445E-06	1.672E-06	1.917E-06	2.123E-06	2.247E-06	1.129E-06
30	3.360E-07	4.103E-07	5.009E-07	6.128E-07	7.212E-07	8.587E-07	1.023E-06	1.193E-06	1.416E-06	1.415E-06

After updating the intersection flows to obtain a new set of $Q_{i,i+1}$ the updated transfer matrix is obtained from (20).

Steady State Stability of the Transfer Matrix:

Definition: Let a_{ij} be any element of A and let a'_{ij} be the corresponding element of A' (updated transfer matrix). Stability of A (or A') implies that for all i,j $|a'_{ij}-a_{ij}| \leq \delta/d_{max}$ and $\delta \geq 0$.

In the above definition δ is an arbitrary parameter chosen to reflect the accuracy of calculation of DO changes and d_{max} is some arbitrary BOD load. To motivate the definition let $\epsilon_{ij} = a'_{ij}-a_{ij}$. Intuitively we would like stability to imply that ϵ_{ij} is so small that changes in DO in section i are negligible when some maximum BOD load, d_{max}, is applied to section j. Therefore, we want

(24) $$|\epsilon_{ij}\, d_{max}| \leq \delta$$

Since $d_{max}>0$, $\delta \geq 0$, and $\epsilon_{ij} = a'_{ij}-a_{ij}$ we have

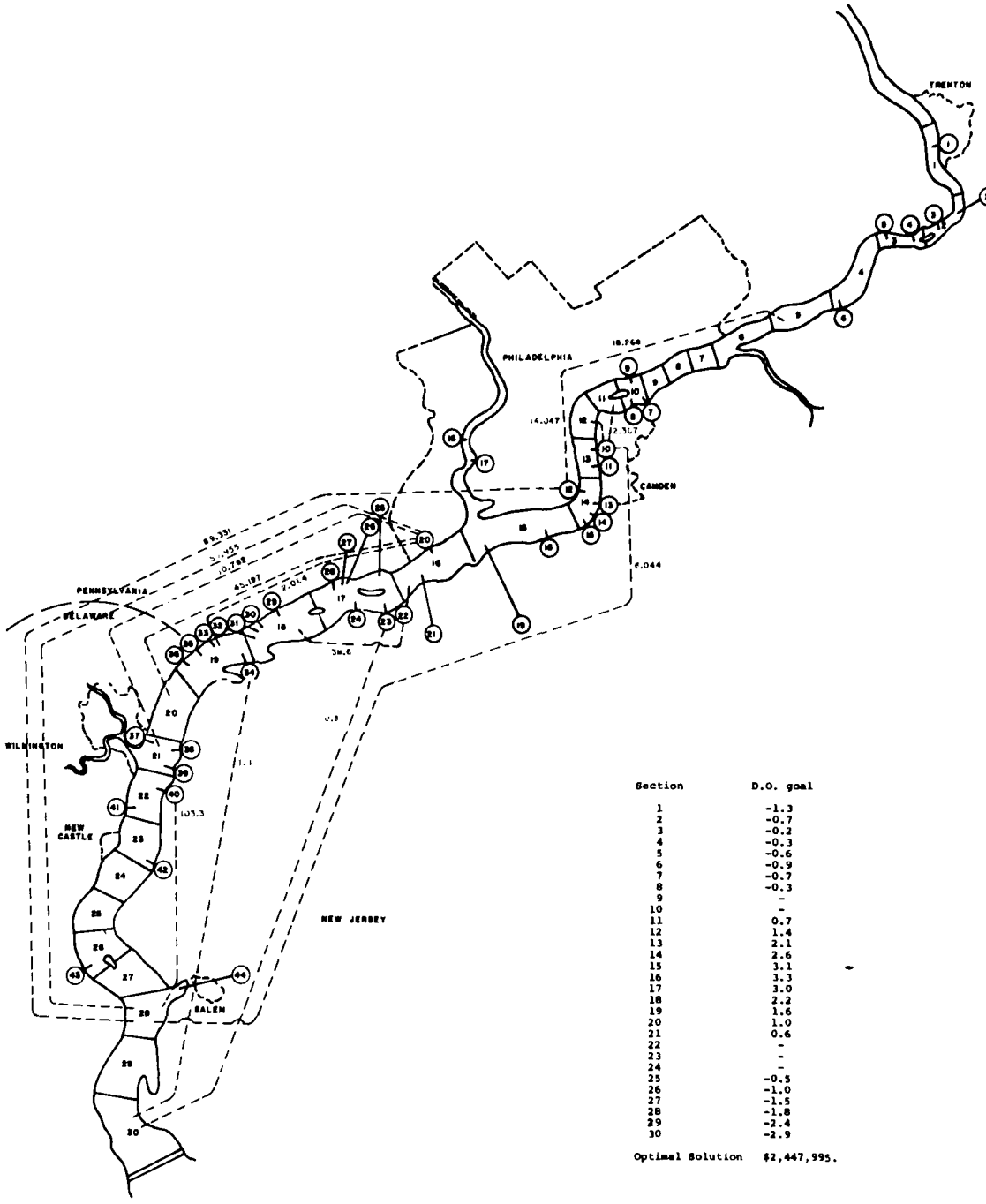
(25) $$|\epsilon_{ij}| = |a'_{ij}-a_{ij}| \leq \delta/d_{max} \quad \text{for all } i,j$$

In practive we generally take d_{max} equal to the maximum polluter input in all sections.

Experimental Results:

The ability to calculate the transfer matrix when needed adds quite a bit of power and flexibility to the water quality control algorithm. The first problem we investigated was to resolve the by-pass piping problem using the new transfer matrix given in Table 5. Recall that this case has linear constraints. A solution of the by-pass piping problem using the transfer matrix of Table 6 can be found in [5] (see Figure 8 page 31). The new solution, with no updating of the transfer matrix, is plotted in Figure 4 in a manner similar to that of Figure 8 in reference [5]. Upon comparison of the two solutions we note that: a) the overall flow patterns are similar; and b) the final costs after eliminating duplicate piping are almost the same ($2,089,148. as compared to $2,070,150. see Figure 9 of [5]). Note that as we are now applying the piping cost equation the unadjusted solution of $2,447,995. is more indicative of the cost with consolidated piping.

The next problem we investigated was to update the transfer matrix and continue the solution carrying each by-pass piping problem to the optimum. The optimal solutions obtained were



Section	D.O. goal
1	-1.3
2	-0.7
3	-0.2
4	-0.3
5	-0.6
6	-0.9
7	-0.7
8	-0.3
9	-
10	-
11	0.7
12	1.4
13	2.1
14	2.6
15	3.1
16	3.3
17	3.0
18	2.2
19	1.6
20	1.0
21	0.6
22	-
23	-
24	-
25	-0.5
26	-1.0
27	-1.5
28	-1.8
29	-2.4
30	-2.9
Optimal Solution	$2,447,995.

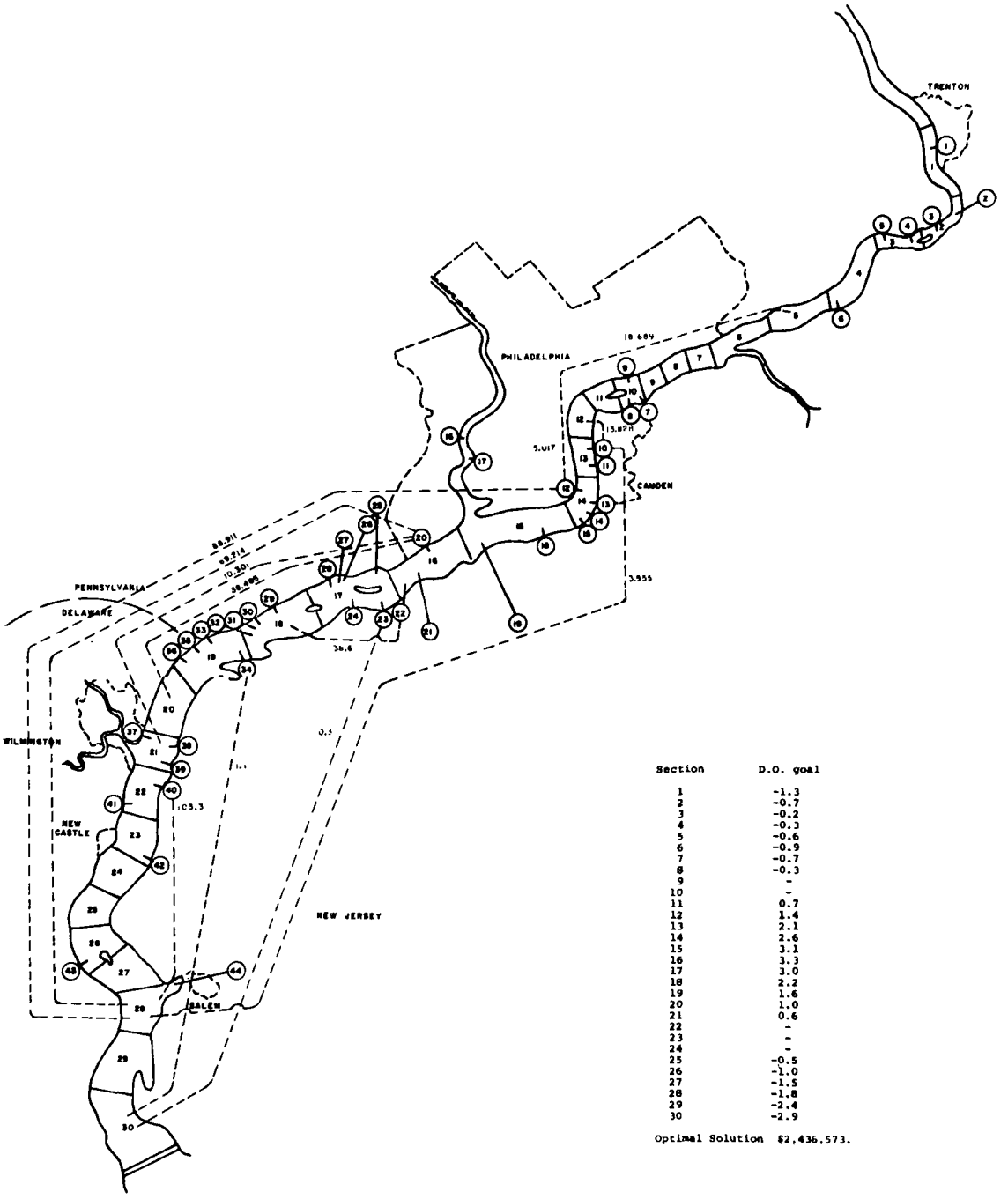
FIGURE 4

SOLUTION OF BY-PASS PIPING PROBLEM
USING NEW FIXED TRANSFER MATRIX

transfer matrix	cost
(Table 5)	$2,447,995.
1st update	$2,436,462.
2nd update	$2,436,573.
3rd update (stable)	

The total computer running time was 132 seconds with one restart at 60 seconds (some calculating time was lost since the restart occurred while solving a by-pass piping problem).

The final solution is plotted in Figure 5. We can conclude that for the by-pass piping problem on the Delaware Estuary updating the transfer matrix is of little or no importance. Since pure by-pass piping is the most extreme case of rearranging the flow pattern the above conclusion holds for any problem configuration.



Section	D.O. goal
1	-1.3
2	-0.7
3	-0.2
4	-0.3
5	-0.6
6	-0.9
7	-0.7
8	-0.3
9	-
10	-
11	0.7
12	1.4
13	2.1
14	2.6
15	3.1
16	3.3
17	3.0
18	2.2
19	1.6
20	1.0
21	0.6
22	-
23	-
24	-
25	-0.5
26	-1.0
27	-1.5
28	-1.8
29	-2.4
30	-2.9

Optimal Solution $2,436,573.

FIGURE 5

SOLUTION OF BY PASS PIPING PROBLEM
USING NEW VARIABLE TRANSFER MATRIX

V. ESTIMATION OF THE DEPTH AND VELOCITY OF A RIVER FOR A GIVEN RATE OF FLOW

Main reference for this study has been: F.M. Henderson, Open Channel Flow (see [6]). No attempt has been made to make specific references and for further clarification regarding the physics of the problem, the reader is referred to the above text or to its equivalent. The motivation for this study stems from the general equation for the behavior of dissolved oxygen in an estuary as suggested by Thomann [10]. One way to improve the DO level is by controlling the flow in the estuary which in turn determines the depth and velocity. The rate of flow appears explicitly in Thomann's equation. The depths are implicit in the volumes of the various sections. If during the solution process of the general model, large changes of flow are generated, the matrix of transfer coefficients has to be updated to correspond to the new flow variables which in turn requires estimation of the new depths resulting from the new rate of flow.

Four cases to be considered in this chapter are:

A. No inflows or outflows.
B. There are spot inflows (pipes).
C. There are spot outflows (pipes).
D. There is a continuous inflow (tributary).

The underlying assuptions in all these cases are:

1. Water level does not vary across any cross-section.
2. Velocities of flow are low (or equivalently--slopes are small) and the steady state velocity is subcritical.
3. Velocity of flow is uniformly distributed in any cross-section.
4. Rectangular cross-section.

Additional assumptions do not apply to all four cases and they are listed in their appropriate sections.

A. Regular flow - fixed rate of flow in the whole section:

The Bernoulli equation:

$$H = y + z + \frac{v^2}{2g}$$

applies to small slopes.

H = total head

y = depth of water in the particular cross-section

v = velocity of flow (average over the cross-section)

z = height of river bed above reference line

g = gravitational constant

For two consecutive cross-sections:

$H_i = H_{i+1} + \ell_{i,i+1}$, $i=0,\ldots,n$ and $i=0$ at the beginning of the first section and $i=n$ at the mouth of the river.

n = number of sections into which the river is divided.

$\ell_{i,i+1}$ = losses between cross-section i and i+1.

Applying the Bernoulli formula to H_i and H_{i+1} gives:

$$y_i + z_i + \frac{v_i^2}{2g} = y_{i+1} + z_{i+1} + \frac{v_{i+1}^2}{2g} + \ell_{i,i+1}$$

Defining: S_i = average slope of river bed at section (i,i+1)

ΔX_{i+1} = length of $(i+1)^{st}$ section

leads to:

$$(z_i - z_{i+1}) = S_i \cdot \Delta X_{i+1} \quad .$$

Substituting gives:

$$S_i \cdot \Delta X_{i+1} + y_i + \frac{v_i^2}{2g} = y_{i+1} + \frac{v_{i+1}^2}{2g} + \ell_{i,i+1} \quad .$$

Eliminating v_i and v_{i+1} by observing that:

$Q = vA = vby$

where:

A = cross-sectional area

b = width of cross-section,

results in:

$$S_i \cdot \Delta X_{i+1} + y_i + \frac{1}{2g} \cdot \left(\frac{Q}{by}\right)_i^2 = y_{i+1} + \frac{1}{2g} \cdot \left(\frac{Q}{by}\right)_{i+1}^2 + \ell_{i,i+1}$$

or:

$$(26)\quad y_i^3 + [S_i \cdot \Delta X_{i+1} - y_{i+1} - \frac{1}{2g}\left(\frac{Q}{by}\right)^2_{i+1} - \ell_{i,i+1}]\, y_i^2 + \frac{1}{2g}\left(\frac{Q}{b}\right)^2_i = 0.$$

<u>The loss function</u>:

$$\ell_{i,i+1} = (S_f)_{\substack{\text{average}\\ \text{over}\\ \text{section } i+1}} \cdot \Delta X_{i+1} + c_L \left(\frac{v^2}{2g}\right)_{\substack{\text{average}\\ \text{over}\\ \text{section } i+1}}$$

where:

S_f = friction slope = $\left(\frac{nv}{1.49}\right)^2 \cdot \left(\frac{1}{R^{4/3}}\right)$

n = Manning's resistance coefficient

R = hydraulic mean radius = A/P

P = wetted perimeter

A = Cross-sectional area

c_L = Eddy coefficient = $2\frac{b}{r_c}$ for angles between 90° and 180°

r_c = radius of curvature of center-line

The arithmatic average used to calculate S_f is a crude approximation but this is also the nature of the whole loss expression. The estimate of c_L is biased upwards severely in most cases, but in the calculation here, the model does not take into account bridges or islands. In addition, the larger the river, the larger the Eddy Coefficient for the same b/r_c.

Substituting gives:

$$\ell_{i,i+1} = \frac{1}{2}\left[\left(\frac{v^2}{R^{4/3}}\right)_i + \left(\frac{v^2}{R^{4/3}}\right)_{i+1}\right]\left(\frac{n}{1.49}\right)^2 \Delta X_{i+1} +$$

$$+ \frac{1}{2}\left(\frac{v_i^2}{2g} + \frac{v_{i+1}^2}{2g}\right) c_{L_{i+1}}$$

Substituting $R = \frac{by}{P} = \frac{by}{b+2y}$ and $v = \frac{Q}{by}$:

$$\ell_{i,i+1} = \frac{1}{2} \cdot \left(\frac{Qn}{1.49}\right)^2 \Delta X_{i+1} \left[\frac{(b_i+2y_i)^{4/3}}{(by)_i^{10/3}} + \right.$$

$$\left. + \frac{(b+2y)_{i+1}^{4/3}}{(by)_{i+1}^{10/3}} \right] + \frac{Q^2 c_{L_{i+1}}}{4g} \left[\frac{1}{(by)_i^2} + \frac{1}{(by)_{i+1}^2} \right]$$

To approximate c_L for angles smaller than 90°:

$$c_L = \frac{2\bar{b}}{r_{i+1}} = \frac{2}{r_{i+1}} \left(\frac{b_i+b_{i+1}}{2}\right) \cdot \frac{\alpha}{90^o} = \frac{b_i+b_{i+1}}{r_{i+1}} \cdot \frac{\alpha}{90^o}$$

where: α = angle of curvature (in degrees).

$$(27)\ \ell_{i,i+1} - \frac{1}{2} \left(\frac{Qn}{1.49}\right)^2 \cdot \Delta X_{i+1} \left[\frac{1}{(by)_i^2} \left(\frac{2}{b_i} + \frac{1}{y_i}\right)^{4/3} + \right.$$

$$\left. + \left(\frac{1}{by}\right)_{i+1}^2 \cdot \left(\frac{2}{b_{i+1}} + \frac{1}{y_{i+1}}\right)^{4/3} \right] -$$

$$- \frac{Q^2 (b_i+b_{i+1})}{4gr_{i,i+1}} \cdot \frac{\alpha}{90} \left[\left(\frac{1}{by}\right)_i^2 + \left(\frac{1}{by}\right)_{i+1}^2 \right] = 0$$

Equations (26) and (27) define a recursive relation between y_i and y_{i+1}. The initial condition required to solve this recursive set of equations is obtained by assuming that at the mouth of the river

$$y_n = \text{CONST.}$$

y_n has to be known to begin the solution process.

Equations (26) and (27) have to be solved by using a numerical method. To update the longitudinal profile after a vector of flow-changes has been induced, there are two possibilites:

a. Identify the new flows in every section and resolve equations (26) and (27).

b. Approximate the changes in the y_i values as $\frac{dy_i}{dQ} \cdot \Delta Q$.

Looking at equations (26) and (27) as components of a vector function which implicitly defines

$$\begin{pmatrix} y_i \\ \ell_{i,i+1} \end{pmatrix} = f(y_{i+1}, Q)$$

and using the implicit function theorem, leads to:

Equation (26) $\quad F_1[(y_{i+1}, Q), (y_i, \ell_{i,i+1})] = 0$

Equation (27) $\quad F_2[(y_{i+1}, Q), (y_i, \ell_{i,i+1})] = 0$

or:

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = 0 = F$$

so that:

$$f'\begin{pmatrix} y_{i+1} \\ Q \end{pmatrix} = -[F'_{y_i, \ell_{i,i+1}}]^{-1} \cdot [F'_{y_{i+1}, Q}] = \begin{bmatrix} \dfrac{\partial y_i}{\partial y_{i+1}} & \dfrac{\partial y_i}{\partial Q} \\ \dfrac{\partial \ell_{i,i+1}}{\partial y_{i+1}} & \dfrac{\partial \ell_{i,i+1}}{\partial Q} \end{bmatrix}$$

Only the first row of the matrix is needed to obtain $\dfrac{dy_i}{dQ}$, since:

$$\frac{dy_i}{dQ} = \frac{\partial y_i}{\partial y_{i+1}} \cdot \frac{dy_{i+1}}{dQ} + \frac{\partial y_i}{\partial Q}$$

$$[F'_{y_i, \ell_{i,i+1}}] = \begin{bmatrix} \dfrac{\partial F_1}{\partial y_i} & \dfrac{\partial F_1}{\partial \ell_{i,i+1}} \\ \dfrac{\partial F_2}{\partial y_i} & \dfrac{\partial F_2}{\partial \ell_{i,i+1}} \end{bmatrix}$$

$$\frac{\partial F_1}{\partial y_i} = \bar{A} = 3y_i^2 + 2y_i \left[S_i \cdot \Delta X_{i+1} - y_{i+1} - \left(\frac{Q}{by}\right)^2_{i+1} \cdot \frac{1}{2g} - \ell_{i,i+1}\right]$$

$$\frac{\partial F_1}{\partial \ell_{i,i+1}} = \bar{B} = -y_i^2$$

$$\bar{C}_i = \frac{\partial F_2}{\partial y_i} = \frac{1}{2}\left(\frac{Qn}{1.49 \cdot b_i}\right)^2 \cdot \Delta X_{i+1} \left[\frac{2}{y_i^3}\left(\frac{2}{b_i} + \frac{1}{y_i}\right)^{4/3} + \right.$$

$$\left. + \frac{1}{y_i^2} \cdot \frac{4}{3}\left(\frac{2}{b_i} + \frac{1}{y_i}\right)^{1/3} \cdot \frac{1}{y_i^2}\right] + \left(\frac{Q}{b_i}\right)^2 \cdot \frac{(b_i + b_{i+1})}{2gr_{i,i+1}} \cdot \frac{\alpha}{90} \cdot \frac{1}{y^3} =$$

$$= \left(\frac{Q \cdot n}{1.49 b_i y_i}\right)^2 \cdot \frac{\Delta X_{i+1}}{y_i} \cdot \left(\frac{2}{b_i} + \frac{1}{y_i}\right)^{1/3}\left(\frac{2}{b_i} + \frac{5}{3y_i}\right) +$$

$$+ \left(\frac{Q}{b_i y_i}\right)^2 \cdot \frac{1}{y_i} \cdot \frac{(b_i + b_{i+1})}{2gr_{i,i+1}} \cdot \frac{\alpha}{90}$$

$$\frac{\partial F_2}{\partial \ell_{i,i+1}} = 1$$

The determinant of this matrix:

$$\Delta = \bar{A} \cdot 1 - \bar{B}\bar{C}_i = \bar{A} + y_i^2 \bar{C}_i$$

Only the first row of $[F'_{y_i, \ell_{i,i+1}}]^{-1}$ is required to obtain the first row of $f'\begin{pmatrix} y_i \\ \ell_{i,i+1} \end{pmatrix}$. This row equals:

$$\left[\left(\frac{\partial F_2}{\partial \ell_{i,i+1}}\right) \cdot \frac{1}{\Delta} \quad \left(\frac{-\partial F_1}{\partial \ell_{i,i+1}}\right) \cdot \frac{1}{\Delta}\right] = \left[\frac{1}{\Delta} \quad \frac{y_i^2}{\Delta}\right]$$

$$[F'_{y_{i+1}, Q}] = \begin{bmatrix} \frac{\partial F_1}{\partial y_{i+1}} & \frac{\partial F_1}{\partial Q} \\ \frac{\partial F_2}{\partial y_{i+1}} & \frac{\partial F_2}{\partial Q} \end{bmatrix}$$

$$\frac{\partial F_1}{\partial y_{i+1}} = \overline{D} = -y_i^2 \left[1 - \frac{1}{g}\left(\frac{Q}{b_{i+1}}\right)^2 \frac{1}{y_{i+1}^3}\right]$$

$$\frac{\partial F_1}{\partial Q} = \overline{E} = \frac{Q}{g}\left[\frac{1}{b_i^2} - \left(\frac{y_i}{b_{i+1} y_{i+1}}\right)^2\right]$$

$$\frac{\partial F_2}{\partial y_{i+1}} = \overline{C}_{i+1}$$

$\overline{C}_{i+1}$ is obtained from $\overline{C}_i$ by interchanging i and i+1 in the y and b values in $\overline{C}_i$.

$$\frac{\partial F_2}{\partial Q} = \overline{G} = \frac{2\,(F_2 - \ell_{i,i+1})}{Q}$$

$$\left[\frac{\partial y_i}{\partial y_{i+1}} \quad \frac{\partial y_i}{\partial Q}\right] = -\left[\frac{1}{\Delta} \quad \frac{y_i^2}{\Delta}\right] \cdot \begin{bmatrix} \overline{D} & \overline{E} \\ \overline{C}_{i+1} & \overline{G} \end{bmatrix} =$$

$$= -\left[\frac{\overline{D} + y_i^2 \overline{C}_{i+1}}{\overline{A} + y_i^2 \overline{C}_i} \qquad \frac{\overline{E} + y_i^2 \overline{G}}{\overline{A} + y_i^2 \overline{C}_i}\right]$$

These expressions may be used to calculate $\frac{dy_i}{dQ} = \frac{\partial y_i}{\partial y_{i+1}} \cdot \frac{dy_{i+1}}{dQ} + \frac{\partial y_i}{\partial Q}$

This is a recursive relation which may generate a large round-off error in addition to a high truncation error which accumulates as i decreases.

It may be both simpler and more accurate to use method a (see p. 40) to update the profile. The same is true in the other three cases examined, as will be shown later.

B. The Longitudinal Profile with Spot Inflows (Pipes):

Additional assumptions for this case:

1. There is a negligible region where the change in heights and velocities spreads across the width of the river and then the cross-sectional water profile is independent of the lateral position.

2. The pipe is perpendicular to the direction of the main flow. This means that the incoming flow does not add momentum in the direction of the main flow.

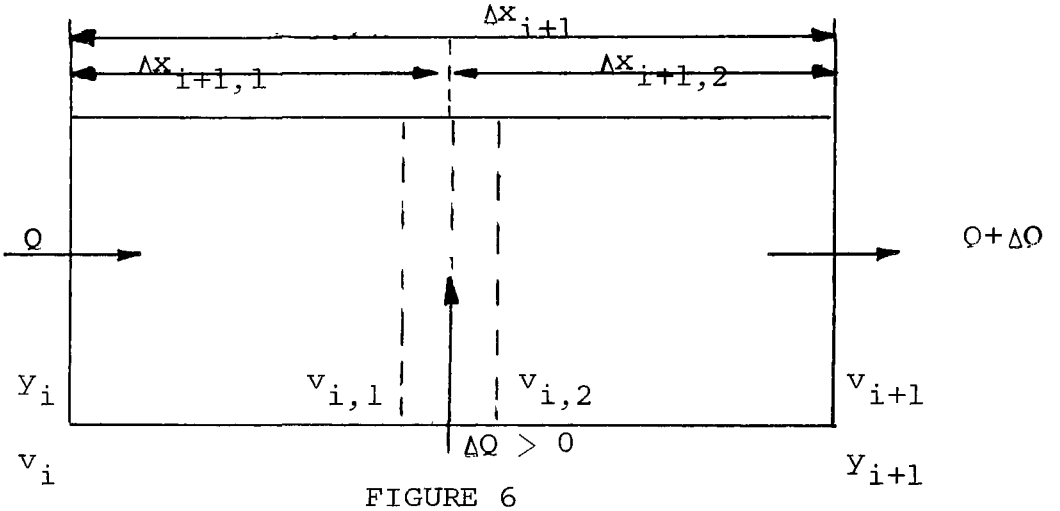


FIGURE 6

SCHEMATIC AND NOTATIONS FOR SPOT INFLOWS

The solving equations is obtained by postulating that the momentum is preserved in the direction of the main flow:

$$Q \cdot v_{i,1} = (Q + \Delta Q) \cdot v_{i,2}$$

$y_{i,2}$ may be calculated recursively from the known situation in cross-section i+1.

$$v_{i,1} = \frac{Q + \Delta Q}{Q} \cdot v_{i,2}$$

ΔQ is assumed to be fixed and positive.

Substituting: $$y_{i,1} = \frac{Q}{b_{i,1} v_{i,1}}$$

$$v_{i,2} = \frac{Q + \Delta Q}{b_{i,2} y_{i,2}}$$

$$b_{i,2} = b_{i,1} \quad ,$$

gives:

$$y_{i,1} = \frac{Q \cdot Q}{b_{i,1}(Q+\Delta Q)\frac{(Q+\Delta Q)}{b_{i,1}y_{i,2}}} = \left(\frac{Q}{Q+\Delta Q}\right)^2 \cdot y_{i,2} = \left(\frac{1}{1+\frac{\Delta Q}{Q}}\right)^2 \cdot y_{i,2}$$

$$\frac{dy_{i,1}}{dQ} = \left(\frac{1}{1+\frac{\Delta Q}{Q}}\right)^2 \left[\frac{2 \cdot \Delta Q \cdot y_{i,2}}{Q^2(1+\frac{\Delta Q}{Q})} + \frac{dy_{i,2}}{dQ}\right] =$$

$$= y_{i,1}\left[\frac{2 \cdot \Delta Q}{Q(Q+\Delta Q)} + \frac{1}{y_{i,2}} \cdot \frac{dy_{i,2}}{dQ}\right]$$

Consider now the case where ΔQ changes. The vector of flow changes will define the new flow in every section. The new dpeths and their first derivatives may be estimated as follows:

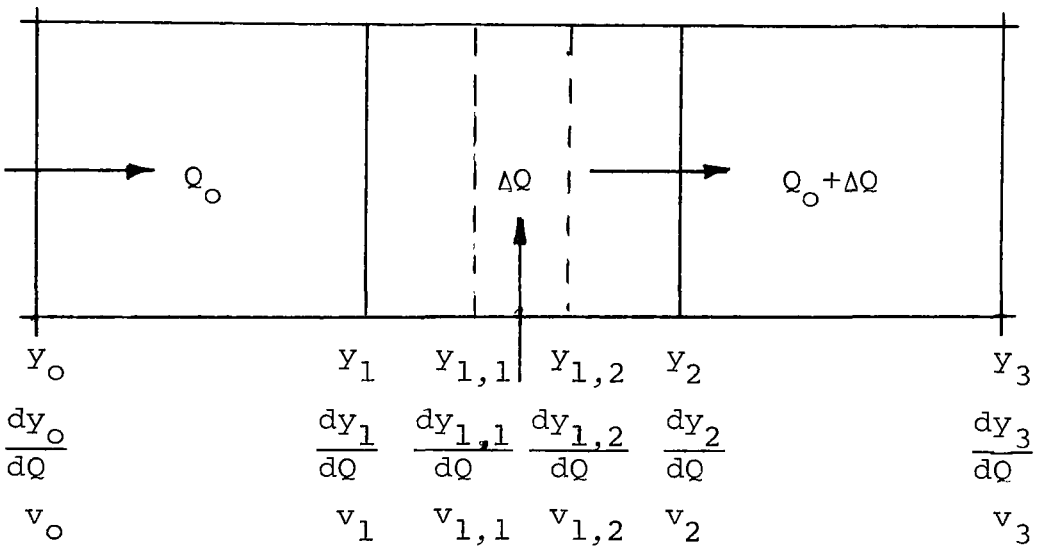


FIGURE 7

SCHEMATIC AND NOTATIONS FOR DERIVATIVES OF SPOT INFLOWS

The situation shown above describes the flow before the change. Suppose a change of $\Delta(\Delta Q)$ occurs. Then for all the cross-sections downstream, the new flow is $Q_0+\Delta Q+\Delta(\Delta Q)$ (assuming as before that $\Delta Q+\Delta(\Delta Q)>0$). For all those downstream cross-sections,

$$\Delta y_i \cong \frac{dy_i}{dQ}\,\Delta(\Delta Q)$$

The new dy/dQ may be estimated from the basic recursive relation. This process of updating will proceed from the mouth of the river up to the cross section next to the transition region ($y_{i,2}$).

For the cross sections upstream from the change:

$$\Delta y_{i,1} \cong \frac{dy_{i,1}}{d(\Delta Q)} \Delta(\Delta Q)$$

$$\frac{dy_{i,1}}{d(\Delta Q)} = y_{i,1} \left[\frac{-2}{Q+\Delta Q} + \frac{1}{y_{i,2}} \cdot \frac{dy_{i,2}}{d(\Delta Q)}\right]$$

Again, since the profile is obtained from a recursive relation starting at the river-mouth, all the sections downstream from the change in flow do not distinguish between the possible sources of the change in flow, so:

$$\frac{dy_{i,2}}{d(\Delta Q)} = \frac{dy_{i,2}}{dQ}$$

and:

$$\Delta y_{i,1} = y_{i,1} \cdot \left[\frac{-2}{Q+\Delta Q} + \frac{1}{y_{i,2}} \cdot \frac{dy_{i,2}}{dQ}\right] \cdot \Delta(\Delta Q)$$

(Note that for computational procedure, the expression of the derivative $dy_{i,1}/d(\Delta Q)$ is similar to the expression $dy_{i,1}/dQ$.)

The new value of $dy_{i,1}/dQ$ may be calculated from the expression for this derivative, substituting all new values which are known.

The rest of the sections upstream may now be updated using $y_{i,1}$, $dy_{i,1}/dQ$ and the basic recursive relation up to the place where the next change occurs.

If there are more than one inflow (or outflow) into the same section and they may not be lumped together, the section may be further broken into sub-sections. A convenient notation procedure may be: All odd second indices are upstream and all even ones are downstream from the location of discharge.

C. The Longitudinal Profile with Spot Outflows (Pipes):

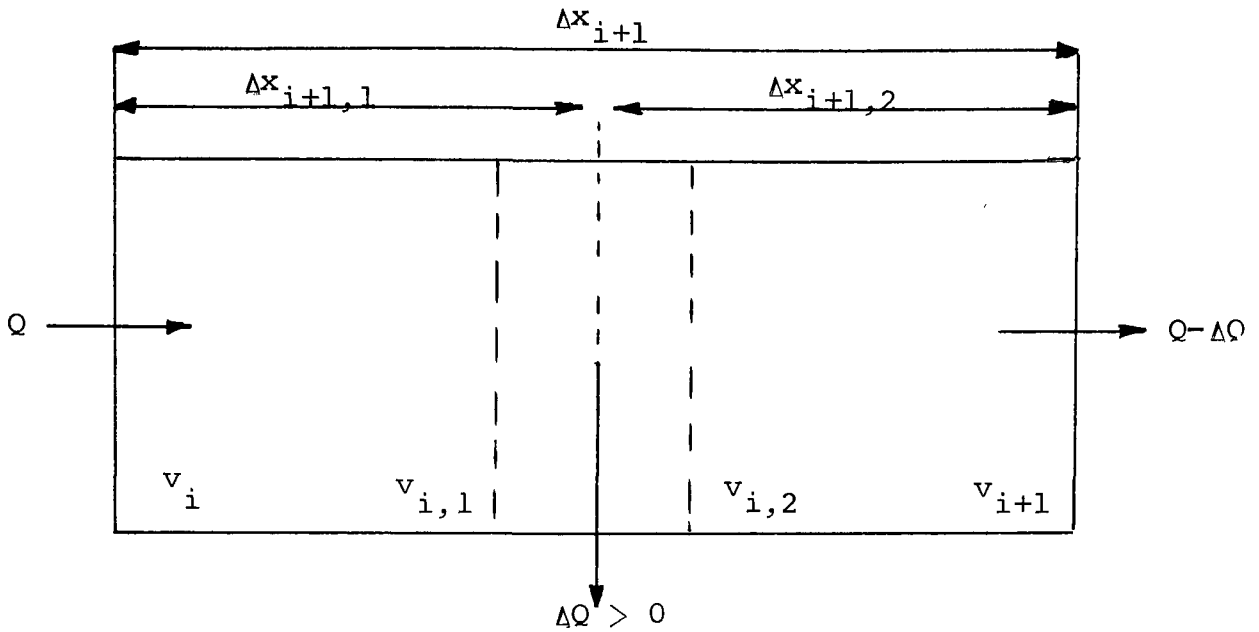


FIGURE 8

SCHEMATIC AND NOTATIONS FOR SPOT OUTFLOWS

In this case, the specific energy (energy per unit of weight, as represented by the Bernoulli equation) is preserved provided some condition is met, as shown later.

Assuming that preservation of specific energy is possible:

$$y_{i,2} + \frac{v_{i,2}^2}{2g} = y_{i,1} + \frac{v_{i,1}^2}{2g}$$

$$v_{i,2} = \frac{Q-\Delta Q}{by_{i,2}} \quad , \quad \Delta Q < Q$$

$$v_{i,1} = \frac{Q}{by_{i,1}}$$

$$y_{i,1} = y_{i,2} + \frac{1}{2gb^2} \left[\left(\frac{Q-\Delta Q}{y_{i,2}}\right)^2 - \left(\frac{Q}{y_{i,1}}\right)^2 \right]$$

$$y_{i,1}^3 - \left[y_{i,2} + \frac{1}{2g} \left(\frac{Q-\Delta Q}{by_{i,2}}\right)^2\right] y_{i,1}^2 + \frac{1}{2g} \left(\frac{Q}{b}\right)^2 = 0$$

$y_{i,2}$ is known from the recursive process.

It will be shown that there are two possible situations:

1. There are three real solutions to the cubic equation , one of them negative. Since the flow is assumed to be sub-critical, the larger positive solution is the appropriate one. The sign properties of the roots of the equation may be obtained directly from the form of the equation. Yet, since it has to be solved anyhow, the nature of the solutions will be demonstrated by evaluating them.

2. Critical conditions develop before the location of change.

To develop the two situations, the starting point is the solution of Situation 1.

The Cardan method [13] for solving the cubic equation is used. Accordingly, the following substitution:

$$y_{i,1} = \mu + \frac{1}{3}\left[y_{i,2} + \frac{1}{2g}\left(\frac{Q-\Delta Q}{by_{i,2}}\right)^2\right]$$

yields:

$$\mu^3 - \frac{1}{3}\left[y_{i,2} + \frac{1}{2g}\left(\frac{Q-\Delta Q}{by_{i,2}}\right)^2\right]^2 \mu + \frac{1}{2g}\left(\frac{Q}{b}\right)^2 - \frac{2}{27}\left[y_{i,2} + \frac{1}{2g}\left(\frac{Q-\Delta Q}{by_{i,2}}\right)^2\right]^3 = 0$$

or:

$$\mu^3 - \frac{c^2}{3}\mu + D - \frac{2}{27}c^3 = 0 \quad .$$

To obtain the solution:

$$A = \frac{c^3}{27} - \frac{D}{2} + \sqrt{\left(\frac{D}{2} - \frac{c^3}{27}\right)^2 - \frac{c^6}{27^2}}$$

$$B = \frac{c^3}{27} - \frac{D}{2} - \sqrt{\left(\frac{D}{2} - \frac{c^3}{27}\right)^2 - \frac{c^6}{27^2}}$$

or:

$$A = \frac{c^3}{27} - \frac{D}{2} + \frac{c^3}{27}\sqrt{\left(1 - \frac{27D}{2c^3}\right)^2 - 1}$$

$$B = \frac{c^3}{27} - \frac{D}{2} - \frac{c^3}{27}\sqrt{\left(1 - \frac{27D}{2c^3}\right)^2 - 1}$$

To get all three roots to be real, it is required that A and B be imaginary, or, the term inside the square-root sign has to be non-positive:

$$(1 - \frac{27D}{2c^3})^2 \leq 1$$

$$-1 \leq 1 - \frac{27D}{2c^3} \leq 1$$

or:

1. $\frac{27D}{2c^3} \geq 0$

Inspection of the equation shows that:

$$D > 0$$
$$c > 0$$

and so $\frac{27D}{2c^3} > 0$ is satisfied.

2. $\frac{27D}{2c^3} \leq 2$

Substituting the original values for D and c:

$$\frac{1}{2g} (\frac{Q}{b})^2 = D$$

$$c = y_{i,2} + \frac{1}{2g} (\frac{Q-\Delta Q}{by_{i,2}})^2$$

or,

$$c = y_{i,2} + \frac{1}{2g} v_{i,2}^2 = y_{i,2}(1 + \frac{1}{2g} \frac{v_{i,2}^2}{y_{i,2}})$$

Substituting, the problem is now transformed into:

$$\frac{4}{27} y_{i,2}^3 [1 + \frac{v_{i,2}^2}{2gy_{i,2}}]^3 \overset{?}{\geq} \frac{1}{2g} (\frac{Q}{b})^2$$

When this condition is met, Situation 1 describes the case. Namely: Ignoring losses in a transition area, the specific energy downstream from the change may also be maintained upstream. The diagram shows the depth-specific energy relation for different levels of specific flow. (The specific flow is the flow per unit of width.)

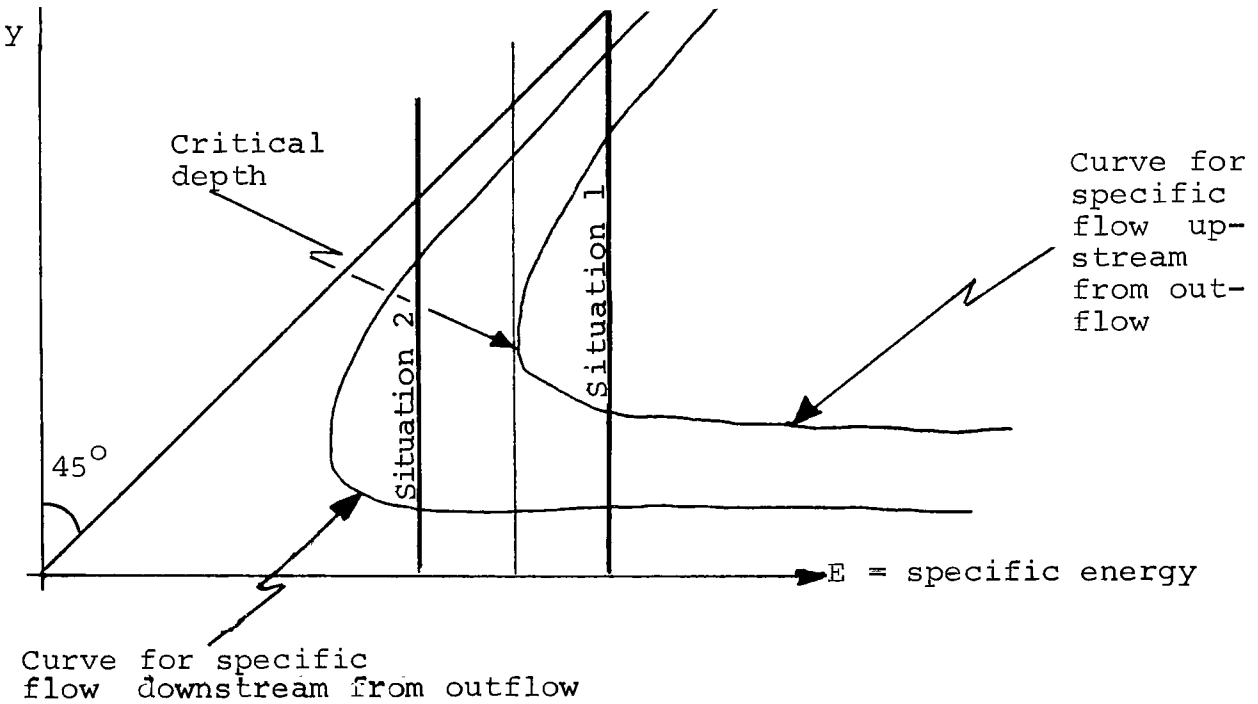


FIGURE 9

CRITICAL, SUPER-CRITICAL AND SUB-CRITICAL FLOWS AS DEPICTED BY THE DEPTH-SPECIFIC ENERGY RELATION

If the change in flow is large enough and $y_{i,2}$ is close enough to the critical value, Situation 2 will prevail. In this case, the specific energies cannot be equal because the specific flow upstream from the location of the change is too high to be able to ever be in the energy-level of the lower flow. Since $y_{i,1}$ and $y_{i,2}$ are determined freely (they are not forced by any controlling device), $y_{i,2}$ will be determined by the energy required to move the specified flow and to overcome losses (namely, the basic recursive relation) and $y_{i,1}$ will determine the lowest energy level possible for the specific flow Q/b. This lowest energy is the critical one. So, in Situation 2, critical conditions will develop upstream from the change and there will be energy dissipation at the transition area so that the conditions defined by the basic recursive equation prevail after the transition. The flow above the critical cross-section will again be sub-critical.

Solution of Situation 1:

$$A = \frac{c^3}{27}\left(1 - \frac{27D}{2c^3}\right) + \frac{c^3}{27}\sqrt{\left(1 - \frac{27D}{2c^3}\right)^2 - 1} =$$

$$= \frac{c^3}{27}\left(1 - \frac{27D}{2c^3}\right) + i\,\frac{c^3}{27}\sqrt{1 - \left(1 - \frac{27D}{2c^3}\right)^2}$$

$$B = \frac{c^3}{27}\left(1 - \frac{27D}{2c^3}\right) - i\,\frac{c^3}{27}\sqrt{1 - \left(1 - \frac{27D}{2c^3}\right)^2}$$

$$A = \frac{c^3}{27}\sqrt{\left(1 - \frac{27D}{2c^3}\right)^2 + 1 - \left(1 - \frac{27D}{2c^3}\right)^2} \cdot \ell^{i\sigma} =$$

$$= \frac{c^3}{27}\,\ell^{i\sigma}$$

Similarly:

$$B = \frac{c^3}{27}\,\bar{\ell}^{i\sigma}$$

where:

$$\sigma = \operatorname{arc\,tg}\left[\frac{\sqrt{1 - \left(1 - \frac{27D}{2c^3}\right)^2}}{1 - \frac{27D}{2c^3}}\right] = \operatorname{arc\,tg}\sqrt{\frac{1}{\left(1 - \frac{27D}{2c^3}\right)^2} - 1}\,,$$

for $1 \neq \frac{27D}{2c^3}$.

$$\mu_1 = \sqrt[3]{A} + \sqrt[3]{B} = \frac{c}{3}\left(\ell^{\frac{i\sigma}{3}} + \ell^{\frac{-i\sigma}{3}}\right) = \frac{2}{3}c\cos\frac{\sigma}{3}$$

$$\mu_2 = -\frac{\mu_1}{2} + \frac{i\sqrt{3}}{2}\left(\sqrt[3]{A} - \sqrt[3]{B}\right) =$$

$$= -\frac{c}{3}\cos\frac{\sigma}{3} + \frac{i\sqrt{3}}{2}\left(i2\,\frac{c}{3}\sin\frac{\sigma}{3}\right) =$$

$$= -\frac{c}{3}\cos\frac{\sigma}{3} - \frac{\sqrt{3}}{3}\,c\sin\frac{\sigma}{3} =$$

$$= -\frac{2}{3}\,c\sin\left(\frac{\sigma}{3} + \beta\right)$$

$$\beta = \arctan\frac{\frac{c}{3}}{\sqrt{3}\,c/3} = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

So:

$$\mu_2 = -\frac{2}{3}\, c \sin\left(\frac{\sigma}{3} + \frac{\pi}{6}\right)$$

Similarly:

$$\mu_3 = \frac{2}{3}\, c \sin\left(\frac{\sigma}{3} - \frac{\pi}{6}\right)$$

To determine the behavior of the roots, σ is explored:

$$\sigma = \arctan\left[\frac{\sqrt{1 - \left(1 - \frac{27D}{2c^3}\right)^2}}{1 - \frac{27D}{2c^3}}\right]$$

For $\frac{27D}{2c^3} = 1$, the value of σ is determined from the original expression for A and B. In that case:

$$A = i\,\frac{c^3}{27} = \frac{c^3}{27}\, \ell^{i\frac{\pi}{2}}$$

$$B = -i\,\frac{c^3}{27} = \frac{c^3}{27}\, \ell^{-i\frac{\pi}{2}}$$

So, $\sigma = \frac{\pi}{2}$ for $\frac{27D}{2c^3} = 1$.

Since $-1 \leq 1 - \frac{27D}{2c^3} < 1$, $0 < \sigma \leq \pi$

the above expression for σ shows that

$$0 < \sigma \leq \pi$$

Substituting gives:

$$0 < \frac{c}{3} \leq \mu_1 < \frac{2}{3}\, c$$

$$-\frac{2}{3}\, c \leq \mu_2 < -\frac{c}{3} < 0$$

$$0 > -\frac{c}{3} < \mu_3 \leq \frac{c}{3} > 0$$

There is also ordering as follows:

$$\mu_1 \geq \mu_3 > \mu_2$$

Substituting

$$y_{i,1} = \mu + \frac{c}{3}$$

yields:

$$\frac{2}{3}c \leq (y_{i,1})_1 < c$$

$$-\frac{c}{3} \leq (y_{i,1})_2 < 0$$

$$0 < (y_{i,1})_3 \leq \frac{2}{3}c$$

Since the flow is assumed to be subcritical, $(y_{i,1})_1$ is always chosen.

$$y_{i,1} = \frac{c}{3}\left(2\cos\frac{\sigma}{3} + 1\right) =$$

$$= \frac{1}{3}\left[y_{i,2} + \frac{1}{2g}\left(\frac{Q-\Delta Q}{by_{i,2}}\right)^2\right] \cdot \left\{2\cos\left[\frac{1}{3}\arctan\sqrt{\frac{1}{\left(1-\frac{27D}{2c^3}\right)^2} - 1}\right] + 1\right\}$$

where:

$$\frac{27D}{2c^3} = \frac{27}{2} \cdot \frac{1}{2g} \cdot \left(\frac{Q}{b}\right)^2 \cdot \frac{1}{\left[y_{i,2} + \frac{1}{2g}\left(\frac{Q-\Delta Q}{by_{i,2}}\right)^2\right]^3}$$

$$\frac{dy_{i,1}}{dQ} = \frac{1}{3}\left\{\frac{dc}{dQ}\left(2\cos\frac{\sigma}{3} + 1\right) + \frac{2c}{3}\left[-\sin\frac{\sigma}{3} \cdot \frac{d\sigma}{dQ}\right]\right\}$$

$$c = y_{i,2} + \frac{1}{2g}\left(\frac{Q-\Delta Q}{by_{i,2}}\right)^2$$

$$\frac{dc}{dQ} = \frac{dy_{i,2}}{dQ} + \frac{1}{2gb^2}\left[\frac{y_{i,2}^2 \cdot 2(Q-\Delta Q) - (Q-\Delta Q)^2 \frac{dy_{i,2}}{dQ} \cdot 2y_{i,2}}{y_{i,2}^4}\right]$$

$$\frac{dc}{dQ} = \frac{dy_{i,2}}{dQ} - \frac{(Q-\Delta Q)}{g(by_{i,2})^2}\left[-1 + \frac{Q-\Delta Q}{y_{i,2}} \cdot \frac{dy_{i,2}}{dQ}\right]$$

$$\frac{d\sigma}{dQ} = \frac{1}{\frac{1}{(1-\frac{27D}{2c^3})^2} - 1 + 1} \cdot \frac{d}{dQ}\left[\sqrt{\frac{1}{(1-\frac{27D}{2c^3})^2} - 1}\right] =$$

$$= (1 - \frac{27D}{2c^3})^2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{1}{(1-\frac{27D}{2c^3})^2} - 1}} \cdot \frac{d}{dQ}\left[\frac{1}{1-\frac{27D}{2c^3}}\right]^2 =$$

$$= (1 - \frac{27D}{2c^3})^2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{1}{(1-\frac{27D}{2c^3})^2} - 1}} (-2) \frac{1}{(1-\frac{27D}{2c^3})^3} \quad (-\frac{27}{2}) \frac{d}{dQ} (\frac{D}{c^3})$$

$$= \frac{27}{2} \cdot \frac{1}{\sqrt{1 - (1-\frac{27D}{2c^3})^2}} \cdot \left[\frac{c^3 \frac{dD}{dQ} - 3Dc^2 \frac{dc}{dQ}}{c^6}\right] =$$

$$= \frac{27}{2c^3} \cdot \frac{1}{\sqrt{1 - (1-\frac{27D}{2c^3})^2}} \left[\frac{dD}{dQ} - \frac{3D}{c} \cdot \frac{dc}{dQ}\right]$$

$\frac{dc}{dQ}$ has been found before.

$$\frac{dD}{dQ} = \frac{Q}{gb^2}$$

Substituting these expressions gives $dy_{i,1}/dQ$.

As mentioned in Case B, changes in ΔQ will be regarded as changes in Q in the sections downstream from the location of change. The only variables affected by a change in ΔQ are $y_{i,1}$ and $dy_{i,1}/dQ$. $y_{i,1}$ may be updated using a first order approximation, for which $dy_{i,1}/d(\Delta Q)$ is necessary:

$$\frac{dy_{i,1}}{d(\Delta Q)} = \frac{1}{3}\left[\frac{dc}{d(\Delta Q)} (2 \cos \frac{\sigma}{3} + 1) - \frac{2}{3} c \sin \frac{\sigma}{3} \cdot \frac{d\sigma}{d(\Delta Q)}\right]$$

$$\frac{dc}{d(\Delta Q)} = \frac{dy_{i,2}}{d(\Delta Q)} + \frac{1}{2gb^2}\left[\frac{-y_{i,2}^2 \; 2(Q-\Delta Q) - (Q-\Delta Q)^2 2y_{i,2} \frac{dy_{i,2}}{d(\Delta Q)}}{y_{i,2}^4}\right]$$

$$= \frac{dy_{i,2}}{d\Delta Q} - \frac{(Q-\Delta Q)}{g(by_{i,2})^2}\left[1 + \frac{Q-\Delta Q}{y_{i,2}} \cdot \frac{dy_{i,2}}{d(\Delta Q)}\right]$$

Substituting:

$$\frac{dy_{i,2}}{d(\Delta Q)} = \frac{dy_{i,2}}{dQ}$$

yields for $\frac{dc}{d(\Delta Q)}$ an expression very similar to that for $\frac{dc}{dQ}$.

Appropriate substitution yields:

$$\frac{dc}{d(\Delta Q)} = \frac{dc}{dQ} - \frac{2(Q-\Delta Q)}{g(by_{i,2})^2}$$

$$\frac{d\sigma}{d(\Delta Q)} = \frac{27}{2} \cdot \frac{1}{\sqrt{1 - (1-\frac{27D}{2c^3})^2}} \left[\frac{c^3 \frac{dD}{d(\Delta Q)} - 3Dc^2 \frac{dc}{d(\Delta Q)}}{c^6}\right]$$

$$= \frac{27}{2c^3} \cdot \frac{1}{\sqrt{1 - (1-\frac{27D}{2c^3})^2}} \left[\frac{dD}{d(\Delta Q)} - \frac{3D}{c} \frac{dc}{d(\Delta Q)}\right]$$

$\frac{dc}{d(\Delta Q)}$ has been found before.

$$D = \frac{1}{2g} \left(\frac{Q}{b}\right)^2$$

$$\frac{dD}{d(\Delta Q)} = 0$$

So:

$$\frac{d\sigma}{d(\Delta Q)} = \frac{27}{2c^3} \cdot \frac{1}{\sqrt{1 - (1-\frac{27D}{2c^3})^2}} \left[-\frac{3D}{c} \frac{dc}{d(\Delta Q)}\right]$$

Substituting into $\frac{dy_{i,2}}{d(\Delta Q)}$ gives a value for the first derivative which may be used to update $y_{i,2}$.

Situation 2:

The critical depth is:

$$y_{i,1} = \sqrt[3]{\frac{1}{g}\left(\frac{Q}{b}\right)^2}$$

$$\frac{dy_{i,1}}{d(\Delta Q)} = 0 \ .$$

Transition between Situations 1 and 2:

If the expression:

$$(28) \quad \{\frac{4}{27} [y_{i,2} + \frac{1}{2g} (\frac{Q}{b})^2 (\frac{1-\frac{\Delta Q}{Q}}{y_{i,2}})^2]^3 - \frac{1}{2g}(\frac{Q}{b})^2\}$$

changes sign as a result of the change in ΔQ, then a transition between the two situations occurs.

If expression (28) changes from negative to positive, then $y_{i,1}$ and $dy_{i,1}/dQ$ are calculated as in Situation 1 with the new ΔQ.

If expression (28) changes from positive to negative, then regardless of $y_{i,2}$, $y_{i,1}$ is critical and $dy_{i,1}/d\Delta Q = 0$. If ΔQ becomes negative, the outflow becomes an inflow. Then the updating has to be done in two steps:

1. To the situation where $\Delta Q = 0$;
2. Using the routine described in Case B.

The procedure is similar in the case of changing an inflow into an outflow.

D. <u>The Longitudinal Profile with Continuous Inflow (Tributaries)</u>:

The notational convention is:

Capital letters - for main stream

Lower case letters - for tributary

Q,q = rates of flow

V,v = velocities

M = total momentum

M_x = component of total momentum in direction of main flow

B,b = widths of main flow and tributary

α = angle between main flow and tributary at intersection.

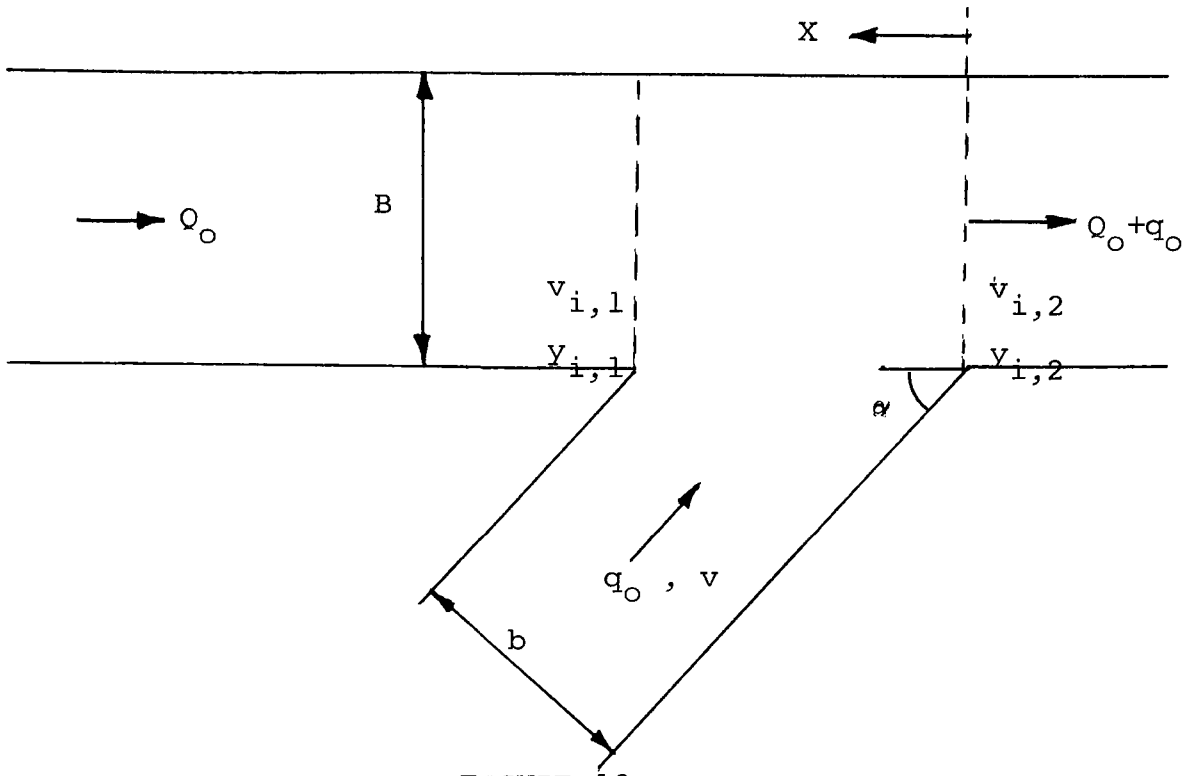


FIGURE 10

SCHEMATIC AND NOTATIONS FOR CONTINUOUS INFLOW

Assumptions:

1. Transition regions have negligible dimensions and effects.

2. The depth of the tributary equals that of the main flow and it is approximated by $y_{i,2}$.

3. The component of momentum at the direction of the main flow is maintained. (Actually this may work only for low velocities. In reality, preserving the momentum would have created a path of flow which is forced to follow the path of the main-stream river-bed. In that case, there is a bending phenomenon which may tend to preserve the velocity. There is little theoretical or empirical knowledge describing the behavior of the water in the junction. In the case of high velocities the depth of the water is not uniform across the river. In addition, there are energy losses which cannot be approximated.)

4. Both flow velocities are sub-critical.

$$M_x = QV + qv \cos \alpha$$

$$Q = Q_o + q_o (1 - \frac{X \sin \alpha}{b}), \quad 0 \leq X \leq \frac{b}{\sin\alpha}$$

$$q = \frac{q_o X}{b} \sin \alpha \quad , \quad 0 \leq X \leq \frac{b}{\sin\alpha}$$

$$v = \text{Const.} = \frac{q_0}{by} \cong \frac{q_0}{by_{i,2}}$$

$$V = \frac{Q}{BY}$$

$$M_x = \frac{Q^2}{BY} + \frac{q_o^2 X}{b^2 y_{i,2}} \sin\alpha \cos\alpha = \frac{Q^2}{BY} + (\frac{q_o}{b})^2 \frac{X}{2y_{i,2}} \sin 2\alpha$$

$$\frac{dM_x}{dX} = \frac{2QY\frac{dQ}{dX} - Q^2 \frac{dY}{dX}}{BY^2} + (\frac{q_o}{b})^2 \frac{\sin 2\alpha}{2y_{i,2}}$$

$$\frac{dM_x}{dX} = A(S_o - S_f)$$

where:

$A = BY$ = cross-sectional area of main flow corresponding to point X

S_o = river-bed slope

S_f = frictional slope

$$S_f = (\frac{n}{1.49})^2 \cdot \frac{V^2}{R^{4/3}} = (\frac{nQ}{1.49 \cdot B \cdot Y})^2 \cdot (\frac{B + Y}{BY})^{4/3} =$$

$$= (\frac{nQ}{1.49BY})^2 \cdot (\frac{1}{Y} + \frac{1}{B})^{4/3}$$

where n is Manning's resistance coefficient

$$\frac{dQ}{dX} = - \frac{q_o}{b} \sin \alpha$$

Equating the two expressions for $\frac{dM_x}{dX}$ gives:

$$B^2 Y^3 S_o - (\frac{nQ}{1.49})^2 \; Y \; (\frac{1}{Y} + \frac{1}{B})^{4/3} =$$

$$= - 2QY \frac{q_o}{b} \sin \alpha - Q^2 \frac{dY}{dX} + (\frac{q_o}{b})^2 \cdot \frac{B \sin 2\alpha}{2y_{i,2}} \cdot Y^2$$

$$\frac{dY}{dX} = (\frac{n}{1.49})^2 Y(\frac{1}{Y} + \frac{1}{B})^{4/3} + (\frac{q_o}{bQ})^2 \cdot \frac{B \sin 2\alpha}{2y_{i,2}} \cdot Y^2 -$$

$$- (\frac{B}{Q})^2 S_o Y^3 - \frac{2q_o \sin \alpha}{Qb} Y$$

where:

$$Q = Q_o + \frac{q_o}{b} (b - X \cdot \sin \alpha)$$

The integration is along X for the given value of Q.

If there is a spot inflow or outflow combined with a continuous inflow, the depth is adjusted accordingly and the longitudinal integration resumed for the updated flow.

To obtain $dy_{i,1}/dQ_o$, numerical differentiation has to be used. In order to do this, the integration should be done for various values of Q which requires calculation of the whole profile for various values of Q.

Concluding Remarks:

This model is a first step in simulating the flow characteristics of a river. It points out two basic problems:

1. The system is recursive, which means that for large systems (long rivers) results may lose their meaning for the upstream sections. Only numerical testing of the model may indicate its validity and limitations.

2. There is a large number of additional combinations of basic flow characteristics to be considred, like supercritical velocities in some sections or in tributaries. The cases considered here are likely to be the more important ones from the point of view of water pollution control. The reason is that the DO change in a high velocity section may be neglected.

The following conclusion may be drawn at this stage: Updating the profile using linear approximation is inferior to re-calculating the whole flow profile.

VI. TRANSIENT EFFECTS IN AN ESTUARY

A. Introduction

The transient behavior of the DO resulting from a change in the system is not considered by the approach to DO control which utilizes the transfer coefficient matrix. This study attempts to justify the above approach by demonstrating that under "real world" conditions, the system representing the Delaware Estuary converges essentially monotonically without developing any pathological behavior. It is also demonstrated that the length of the transient period (i.e., the time required to get within $\mp$ 1% of the steady state solution and stay there) in the slower sections is approximately one month.

B. Representation of the Transient

Thomann's model [10] consists essentially of two systems of differential equations, one describing the behavior of B.O.D. and the other one describing behavior of DO. The second system couples with the first one through the B.O.D. levels.

Theoretically, an analytic solution to the system of differential equations is available. Its derivation in this case, however, involves inversion of a 30 x 30 matrix (for the 30 sections of the Delaware) with analytic expressions (as opposed to a matrix with numerical entries). This fact provides a convincing argument for numerical integration. In addition, the result of an analytic solution is again a 30 x 30 matrix which does not have any intuitive meaning, necessitating numerical evaluation of the transient behavior under assumed conditions of loading.

Alternative approaches to the numerical integration are either to investigate the stability of the system (Pole-Zero analysis) or to load it with various patterns of Dirac Deltas (spikes). Both possibilities have been explored by Thomann ([10], p.27, [12], p.354). The main shortcoming of these approaches lies in the fact that they do not provide a simple answer to the question: What would be the consequences of a change in the system and would the transient create undesirable effects?

The approach adopted in this study tries to provide the answer by sampling numerically a few responses under conditions which are plausible in reality. This approach provides also an indication on the sensitivity of the system to changes in certain parameters.

C. Assumptions

The transient reaction of a linear system does not depend on the forcing function (the transient is the homogeneous solution).

However, since a numerical time-pattern of the system's behavior was sought, a simple forcing function which would give intuitive meaning to the result, had to be assumed.

The most obvious one would have been to remove all sources and sinks of oxygen and to evaluate, under proper initial conditions, the mathematical homogeneous solution. The disadvantage of this procedure lies in the fact that the physical phenomenon of saturation with DO is ignored. Since, as explained later (Part D), the system may be allowed to oscillate between the zero-level of DO and the saturation level of DO, the following alternative was preferred:

By removing all sources and sinks of oxygen except for the reaeration from the atmosphere and by assuming that the water above the first section and below the last one is saturated with oxygen, (together with an additional simplifying assumption explained in the sequel), the system is driven to a steady-state solution at the DO-saturation level.

The additional assumption concerns the in-flowing tributaries:

The system of differential equations describes conservation of matter. Accordingly, the amount of flow into a section has to equal the amount of water flowing out. This is easily achieved by equating these two quantities in the computer program, however, this implies that the conditions in the in-flowing tributaries are the same as those prevailing in the next section upstream. This is a somewhat artificial assumption but it is the simplest one that could be made if only the main stream is to be solved for.

It is easily seen that under these assumptions the DO saturation level is the steady state solution for Thomann's equation.

D. Computation Method

As stated before, no analytic solution was sought for the system. Instead, a numerical integration subroutine was used to generate a time pattern of the behavior of DO. The integrating subroutine utilized Hamming's modified predictor-corrector method. This subroutine checks the possible truncation error at each step of the integration and automatically adjusts the size of the interval of integration if the error exceeds a specified value.

The integration period was deliberately chosen as 200 days, which is much longer than the transient period (as it turned out to be), in order to discover beats (dampening of the oscillation and then a renewed increase in the amplitude) if any would have occurred.

It should be stressed that Thomann's major assumption was the

linearity of the main system. This assumption may not be valid when the system approaches its boundary conditions, namely, no DO or saturation, the reason being that at zero DO a different physical process takes over, while the saturation phenomenon is generated by a stochastic process of exchange of oxygen between the atmosphere and the DO. These conditions are not explicit in the system of equations, so there is nothing inherent in the system of equations to prevent the solution from oscillating above the saturation value or below zero if the numerical structure of the equations is appropriate. As a result, one of the basic problems was the adequacy of Thomann's model for describing the transient. This is the reason why it was not clear apriori if a beat would develop, or if the system would try to oscillate beyond its boundary values.

The solution to the second problem was to bound the DO values and their derivatives in case of a bad oscillation. As shown in Part E, the actual runs were designed to investigate the necessity of constraining the DO levels, with the result used to indicate the adequacy of the model in estimating the transient behavior.

E. Computation and Results

1. The conditions assumed correspond to those prevailing in the summer since this is the critical period.

2. The starting conditions for the integration of the DO levels were the average values in the summer of 1964 (Table 3, Schaumburg [9]). The numbers are given in Table 7.

3. The saturation value was calculated as:

$$c_s=14.652-0.410222T+0.0079910T^2+0.00007777T^3$$

(Source -- Pence, Jeglic, Thomann [8]).

Substituting $T=25^{\circ}C\cong 77^{\circ}F$

yields

$$c_s=10.605976 \text{ mg}/\ell$$

So the value used for the integration was 10.6.

4. Since the numerical integration is a sequential process and at every step values are computed sequentially (by a DO-loop) for all sections of the river, starting with 1 and ending at 30, the errors (truncation and round-off) tend to accumulate towards the later points in time and towards the sections indexed by higher numbers.

The integrating subroutine estimates the error for each component of the solution vector generated at each step of the integration. Then the sum of those error terms is compared with a predetermined value. If the error-sum turns out to be too high, the integration interval is halved and the step is repeated.

An option is offered by the subroutine to assign weights to the individual error terms, thus stressing the accuracy of certain components of the solution vector.

Utilizing this option, an attempt was made to improve the results by re-weighting the error terms of the individual sections. The purpose was to increase the sensitivity of the error-sum to those components corresponding to sections of the estuary which were indexed by higher numbers.

The re-weighting was done proportionally to n^2 and to $\sqrt{n}$, where n is the section number. In the case of $\sqrt{n}$, the error term corresponding to section n was multiplied by $\sqrt{n}/\sum_{n=1}^{30}\sqrt{n}$. In the case of n^2, the weight was $n^2/\sum_{n=1}^{30} n^2$.

5. Since the system turned out to be insensitive to the variations tried, they will be listed and the results given for one of them.

   Sensitivities checked:

   i) increasing the saturation value to 12.6 mg/ℓ and decreasing it to 8.2 mg/ℓ.

   ii) forcing as opposed to not forcing the DO levels to be between 0 and the saturation value.

6. As stated before, the results were insensitive to the above variations. Re-weighting as n^2 improved a little the numerical behavior. (For explanation, see comments following Table 7.) Forcing the DO to be at most equal to the saturation value and the derivative to be 0 or lower in case the section is saturated, introduced additional error and the system converged to a somewhat lower value than the saturation level. The runs made without these constraints demonstrated almost no oscillation above the saturation value, indicating that the linearity assumption is adequate in this sense.

7. Lengths of time for the system to converge to $\mp$ 1% of the saturation value are listed in Table 7. In this run the saturation value was 10.6 mg/ℓ, the system was not forced to remain below this value and the error terms were re-weighted as n^2.

Section Number	Transient Period in Days	Starting Conditions mg/ℓ	Error Range mg/ℓ
1	4	7.5	nil
2	4	7.0	"
3	6	6.2	"
4	10	5.1	"
5	14	5.1	"
6	17	5.6	"
7	21	5.4	"
8	24	5.3	"
9	25	3.0	"
10	27	2.2	"
11	29	2.3	"
12	30	1.5	"
13	31	1.2	"
14	32	1.1	"
15	33	1.1	"
16	33	1.0	"
17	33	1.0	"
18	33	1.2	"
19	33	1.4	"
20	32	2.2	11×10^{-5}
21	31	2.9	197×10^{-5}
22	30	3.5	759×10^{-5}
23	31	4.1	2386×10^{-5}
24	29	4.8	7201×10^{-5}
25	28	5.5	13230×10^{-5}
26	25	5.8	15816×10^{-5}
27	22	6.2	13395×10^{-5}
28	23	7.2	6799×10^{-5}
29	21	7.6	743×10^{-5}
30	16	8.0	56×10^{-5}

TABLE 7

TRANSIENT PERIOD OF SECTION i, CORRESPONDING STARTING CONDITIONS AND COMPUTATIONAL ERROR

The error-range (last column in Table 7) is the maximal absolute value of the difference between the following two numbers:

i. The saturation value (10.6 mg/ℓ).

ii. The numbers generated by the computer for the period between the end of the transient phenomenon (column 2 of Table 7) and the end of the integration process (200 days).

These deviations represent the effect of the cummulative error generated by the numerical integration process. Decreasing these deviations was considered as "improving the numerical behavior" (see E.6) and this was the reason for re-weighting the error terms as explained in E.5.

The maximal deviations occur in sections 25, 26 and 27 and they constitute an error of about 1.5%. The deviations in the rest of the sections are well below 1%.

These observations may serve as indications on the accuracy of the numbers generated by the integrating subroutine and on the validity of the results of this section.

VII. POLLUTION TAXES AND CHARGES IN PERSPECTIVE

Previous work [4], presented a model to find a minimal treatment cost solution to a water quality problem. A solution to such a problem gives an efficient solution in terms of cost minimization. However, such a solution may be costly to implement and enforce unless each polluter accepts the regional solution and his share of its costs. We now wish to discuss whether there are decentralized mechanisms so that decisions by individual polluters correspond to minimization of regional costs. The possibility of the decentralization of decision making is important in economics since enforcement costs are minimized when the social optimum can be achieved through individual decision making. In economics, taxes or charges of various types are often suggested as a decentralized means of making individual behavior correspond to a social optimum. Several taxation and charge schemes will be discussed below.

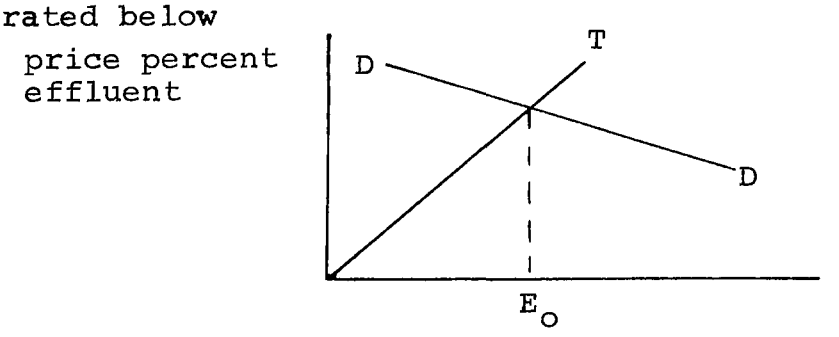
Examination of the literature concerning the economics of water quality reveals the assumption that society has a right to a certain level of water quality, or there is a socially optimum level of water quality. By implication, the polluters should either

1. be forced to treat waste material at a high enough level to maintain the quality required by society

or 2. pay the costs society incurs to maintain the required water quality.

In this view, two types of tax schemes have been discussed, those relating to enforcing a quality standard and those relating to raising revenue. The first type can be called an enforcement tax. One example of such a tax would be to charge a polluter an extremely high tax (or fine) on wastes dumped above a certain level. Thus, above this level, the polluter would rather cease polluting than pay the fine. (To do this, he could either curtail production or build a treatment facility.) Below this level, the polluter is in effect given free rights to pollute. Such a tax would not be intended to raise any revenue.

Another enforcement-type tax would be to have a uniform tax per unit of waste and the tax would be computed so that a polluter would dump no more than a certain amount. This is illustrated below



DD represents a polluter's demand to dump effluent as a function of the tax. If E_o is the amount of effluent to be allowed under a quality standard, then the per unit tax which would result in E_o is given by the slope of T. If the polluter would want to produce at a higher level than that corresponding to E_o, then again he would have to build waste treatment facilities. A related concept would be to view the service of waste disposal via a river as a public resouce to be sold at a price. The price and amount sold is set to ration use of the service corresponding to the quality goals. In both these cases, the tax revenue collected is "gravy" since the tax collecting agency does not construct treatment facilities; the taxes serve only to force achievement of the quality standards. Any treatment undertaken under these enforcement tax systems would be at an individual level and would be individually paid for. Because of the individual emphasis, such tax systems would not result in minimizing regional treatment costs.

The second type of tax scheme would be for the purpose of raising revenue for treatment purposes. For instance, a polluter could be charged for waste disposal according to the social cost of his pollution and the revenues collected could be used either to alleviate the damages or pay for treatment. Here again, this may not lead to the least cost regional solution since it is always easier and cheaper to treat wastes before they enter a stream than to have to remove them from the stream once they are there. Alternatively, revenue could be raised through charging the polluter for his use of a public or "joint" waste treatment facility. It would not be necessary for the charge to be collected by a governmental agency in this case. A contractual agreement could be formed among several polluters to operate a joint facility such as a regional treatment plant or reservoir. The charge for use of the facility would be based on cost. The advantage of this system is that the least cost regional treatment system could be obtained on a voluntary basis. However, in this case, there would first have to be some sort of enforcement mechansim, either legal or through enforcement taxes, to give incentives to polluters to have wastes treated in the joint facility rather than dumping them. If enforcement taxes were used as incentives, then the treatment charges would have to be lower than the tax; otherwise polluters would pay the tax rather than treat their wastes in the regional facility. The charges would also have to be low enough for the polluters to choose the joint treatment alternative rather than treating their own wastes individually. The problem is complicated by the need to cover full costs of building and operating the joint facility from the charges.

It is possible that a proposed tax could fall into both of the above two categories as it affects polluters. For example, suppose a uniform effluent charge of $.05 per mg/d is to be levied on effluent per unit B.O.D. concentration to pay for treatment costs. Because of differing manufacturing processes,

locations and other parameters the cost of removing waste material may vary for polluters. For this reason one polluter may have an on-site B.O.D. removal cost of $.04 per mg/d and another on-site B.O.D. removal cost of $.06 mg/d. In this case, the second polluter would pay his effluent charges since his marginal removal cost is more than the cost of dumping it in the river and the first polluter would treat his own waste. This is not inconsistent with the least cost regional solution since the first polluter can remove his wastes himself more cheaply than in a regional plant; such a situation can arise when a polluter is so far from a regional plant that piping costs overcome possible economies of scale. Here we see that the tax has served two purposes. In the case of the first polluter he has been forced to treat his own wastes. In the case of the second polluter revenue has been raised to defer the cost of treatment undertaken by the "government" or tax collection unit. However, it is clear that unless treatment plant sizes and charges are based on the least cost regional solution, in general the result of such charges may not cover costs of the facilities. That is, in the above example, if a treatment plant were designed to treat wastes of both the first and second polluter and the charge set accordingly at $.05 per mg/d, then the plant would in fact be too large since the first polluter would do his own treatment rather than use the plant at that charge.

The determination of charges for the purpose of quality enforcement is generally based on two criteria:

1. Equity between polluters;

2. quality resulting from implementation of charges meets quality goals.

Johnson in [7] determines several types of effluent charges which meet the second criterion above. The methods used were called least-cost, single effluent charge, and zoned effluent charge.

The least-cost method proposed by Johnson seeks to find the set of effluent charges which will result in the minimization of the sum of the taxation costs incurred by the individual polluters subject to the constraint that the quality goals are met. This approach could possibly result in a different effluent charge for each polluter.

The second approach seeks to impose a single effluent charge in terms of $/lbs. to all polluters, again subject to the constraint that the quality goals are met. Another approach, the zoned method divides the polluters between regions and assigns a single effluent charge to each region to minimize the total taxation cost over the region. Actuarlly, all of these methods are variants of the zoned method since the single charge method treats the entire basin as one zone and the least-cost method in effect

assigns each polluter to a zone. In all cases the least-cost solution is strived for given the constraints on the charges and the quality requirements. Johnson's charges are all examples of enforcement taxes. His cost minimization criterion is a way of determining how polluting rights are to be distributed and is thus an equity criterion. It would also be possible to devise other methods of assigning effluent charges. For example, instead of zones the polluters could be divided with respect to type, each type being assigned an effluent charge. Assigning the same charge for each type gives another equity criterion.

As already mentioned, there is no reason to assume that the revenue obtained from a uniform effluent charge would exactly meet the cost of treatment. Also, an enforcement type tax would not in general result in minimizing basic treatment cost. For these reasons, recently the revenue collection type of tax has become more interesting since the thrust has been to reduce total basin costs significantly through the use of treatment alternatives such as by-pass piping, regional plants and flow augmentation. Each of these alternatives creates problems of joint funding and operation of facilities. In such a case the revenue necessary for the operations of these joint facilities must be obtained from the polluters concerned. Charges of the revenue collection variety are usually based on three criteria:

1. Equity between polluters;

2. quality goals maintained;

3. revenue sufficient to cover costs incurred by tax collecting unit.

Graves, Hatfield, and Whinston [5] construct a pricing scheme in their paper on by-pass piping which is of this type. Their scheme is based on an equity criterion of charging polluters according to damages. The scheme is not self-enforcing and so would have to be preceded by some enforcement mechanism to give incentives to meet the quality goal through by-pass piping. The minimum cost piping arrangements is first solved for subject to meeting the quality goals and conservation of flow constraints. The dual variables associated with the flow conservation constraints measure the marginal cost of extra effluent flow from an individual polluter to the least cost basin solution. Suppose each polluter were charged according to the dual variable per unit effluent. This would lead to a charge of

$$TP_k = \frac{\partial \hat{TC}}{\partial \hat{F}_k^E} \hat{F}_k^E$$

$\hat{F}_k^E$ denotes the optimal effluent flow from polluter k from solution of the least cost programming problem, $\hat{TC}$ is the least cost solution, and the partial derivative is evaluated for the least cost solution. Total revenue from this charge would be

$$\sum_{k=1}^{n} \frac{\partial \hat{TC}}{\partial \hat{F}_k^E} \hat{F}_k^E = \sum_{k=1}^{n} TP_k \tag{29}$$

which in general would not equal TC. The expression in (29) i only guaranteed equal to TC for all values of $\hat{F}_k^E$, k=1,n if the least total cost function is linear homogeneous in the effluent flows.

In the small 5 polluter example in the Graves, Hatfield, Whinston paper [5], the least-cost solution is $55,577. The total revenue collected using an effluent charge equal to the value of the dual variables associated with the flow consistancy constraints is $179,499. In terms of the notation above

$$\hat{TC} = \$55,577$$

$$\sum_{k=1}^{5} TP_k = \$179,499$$

or

$$\hat{TC} < \sum_{k=1}^{5} TP_k \ .$$

In the Graves, Hatfield and Whinston paper the revenue is forced to equal cost by redefining the total payment of polluter j.

$$TP_j^* = (TP_j / \sum_{k=1}^{n} TP_k) \ \hat{TC}$$

This automatically gives

$$\sum_{k=1}^{n} TP_k^* = \hat{TC} \ .$$

This cost allocation is based on achieving the quality goals at least cost, costs are covered from the revenues, and there is an equity criterion. The equity criterion is based on polluters causing the same marginal damage paying the same per unit

effluent charge, where marginal damage is defined to be the increase in the basin solution cost due to an extra unit of effluent.

Although the above is one type of equity criterion, it is clear that there are other types, such as the same type of pollutant, or the same charge for polluters in the same zone. The notion of what is meant by equity needs to be further examined. Equity is included as a criterion for both revenue and enforcement taxes. Its importance is due both to our notions of fairness and to the difficulty of enforcing and collecting the tax or charge when it is not felt to be equitable by those who will be charged. In general, the notion of equity is based on two principles:

1. Polluters who cause more damage should pay more.

2. Polluters who cause equal damage should pay the same.

The problem which is evident from these requirements is that of interpreting "damage". The proposal to charge polluters of the same type the same amount per unit weight makes the implicit assumption that polluters who dump equal weights of effluent of the same composition are causing equal damage. However, this assumption may be false, for instance when the flows (hence concentrations) of their effluents differ. The location of a polluter also determines how much damage he will cause. A polluter located in an area of high water quality (and/or high flow) will do considerably less damage than the same polluter located where the water quality is poor (and stream flow is low). Damage also depends on what is located downstream of a polluter. Damage will be much less if a pulp mill is located downstream than if a municipal water plant is located downstream from a polluter. Under closer examination it is not at all clear that any measure of effluent composition alone reflects pollution damage. In an economic sense, the only measure of damage which is really relevant is the amount of additional cost necessary to maintain desired quality goals due to the existence of polluter k. If polluter k dumps a certain effluent but causes no additional treatment cost to the system (i.e., quality constraints are not violated), then his existence is irrelevant as far as any treatment cost scheme is concerned.

The Graves, Hatfield, and Whinston [5] paper recognized cost in this way as a reflection of damage through their use of the dual variables of the programming model to generate equitable shares of the total basin costs. The problem associated with this measurement of damage, as with any sort of marginal concept, is that this information is only "local" in nature. Since the relationship between the optimal basin solution, and the flows from the individual polluters is non-linear, the marginal cost at the optimum solution of an additional unit of flow from

polluter i is no reflection of the added cost to meet the standards if effluent from a polluter were to be increased by more than one unit. Thus, the equity criterion used in the Graves, Hatfield, Whinston paper is not completely satisfactory.

Incremental Cost Allocation Scheme

In the previous sections of this paper we have indicated several faults associated with the pricing and cost allocation systems presented in the literature. These faults can be summarized as follows:

1. Failure of revenue collected to meet total costs of treatment.

2. Failure of some pricing systems to be based on the most efficient (least-cost) combination of treatment methods.

3. Failure of pricing schemes to satisfy acceptable equity criteria.

In this section we will present a cost allocation system which we feel adequately deals with these problems. This approach takes as given the present composition of polluters in the river basin and the given quality goals and attempts to allocate the costs of a basin wide system. Both the costs of individual polluters and the joint costs of using regional treatment plants are considered in the allocation scheme.

After an optimal solution to a basin polluter problem, such as that described earlier, has been obtained, it is not at all clear what part of the cost of that system of dams, treatment plants, and pipelines can be attributed to the fact that a particular polluter is in the basin. That is, when several polluters are present, it is not clear who has the responsibility for violating the quality constraints and hence the exact damage costs attributable to any one polluter is not well defined. If we could measure the cost that an individual polluter causes to the system, then we would have some basis to decide what his contribution to the total cost of the abatement system should be. We will now describe a way of allocating the total system costs to polluters based on the idea of each polluter paying the incremental cost he causes the system.

Consider a basin with only two polluters designated 1 and 2. Denote the least-cost solution to the basin problem (29) by C(1,2). The incremental cost caused by polluter 1 being in the basin, given that polluter 2 is in the basin, is defined to be

$$IC(1) = C(1,2) - C(2)$$

where C(2) is the cost of polluter 2 treating his own waste.

Both the values of C(1,2) and C(2) can be computed using the programming model in the previous section.

A logical tax based on each polluter paying his social cost would then appear to be to charge polluter 1 IC(1) and polluter 2 IC(2), where

$$IC(2) = C(2).$$

We note that

$$IC(1) + IC(2) = C(1,2)$$

which means that the total cost of the basin project has been covered by this tax scheme. However, the following question now arises: What if the polluters are considered in reverse order? Consider the following definitions of IC*(1) and IC*(2):

$$IC^*(1) = C(1)$$

$$IC^*(2) = C(1,2) - C(1)$$

If polluter 1 is charged IC*(1) and polluter 2 is charged IC*(2) the total cost of the basin system is covered as before, but the charge to each polluter differs in these two schemes. As an example, consider a case where if either of the polluters are alone in the basin the quality goals are met without treatment and if they are both in the basin the goals are violated. Assume that the optimal solution when both are present is to build a reservoir for flow augmentation at a cost of C(1,2) = \$100. The values of C(1) and C(2) are both zero since the standards are met if either polluter is absent. If the first calculation of incremental cost is used the charges for polluters 1 and 2 are:

$$IC(1) = \$100$$

$$IC(2) = \$0$$

If the second calculation is used the charges are:

$$IC^*(1) = \$0$$

$$IC^*(2) = \$100$$

Polluter 1 clearly would prefer the second allocation over the first cost allocation and by the same token polluter 2 would prefer the first allocation. Since either ordering of the polluters is equally plausible, we may average all the possible incremental costs which each polluter could cause to obtain a cost allocation of

$$\rho(1) = \tfrac{1}{2}\, IC(1) + \tfrac{1}{2}\, IC^*(1)$$

$$= \tfrac{1}{2}\,(100) + \tfrac{1}{2}\,(0) = 50$$

and

$$\rho(2) = \tfrac{1}{2}\, IC(2) + \tfrac{1}{2}\, IC^*(2)$$

$$= \tfrac{1}{2}\,(0) + \tfrac{1}{2}\,(100) = 50$$

The sum of the payments from the polluters covers the total basin cost of C(1,2) = \$100. However, the equity problem has been taken care of by considering all possible orders of the polluters, in this case (1 2) and (2 1), and averaging over possible definitions of incremental cost for each polluter.

This approach is easily extended to a case of three polluters where there are six possible orders for calculating incremental costs for each polluter. For example, considering polluter 1, the following orders are possible.

Order 1:	(1 2 3)	polluter 1 before polluters 2 and 3
Order 2:	(1 3 2)	polluter 1 before polluters 2 and 3
Order 3:	(2 1 3)	polluter 1 after polluter 2
Order 4:	(3 1 2)	polluter 1 after polluter 3
Order 5:	(2 3 1)	polluter 1 after polluters 2 and 3
Order 6:	(3 2 1)	polluter 1 after polluters 2 and 3

Corresponding to each of these orders is a way of defining incremental cost for polluter 1 (the subscript denotes the order under consideration):

$$IC_1(1) = C(1)$$

$$IC_2(1) = C(1)$$

$$IC_3(1) = C(1,2) - C(2)$$

$$IC_4(1) = C(1,3) - C(3)$$

$$IC_5(1) = C(1,2,3) - C(2,3)$$

$$IC_6(1) = C(1,2,3) - C(2,3)$$

For polluter 2, the following incremental cost definitions are possible:

$$IC_1(2) = C(1,2) - C(1)$$

$$IC_2(2) = C(1,2,3) - C(1,3)$$

$$IC_3(2) = C(2)$$

$$IC_4(2) = C(1,2,3) - C(1,3)$$

$$IC_5(2) = C(2)$$

$$IC_6(2) = C(2,3) - C(3)$$

And for polluter 3,

$$IC_1(3) = C(1,2,3) - C(1,2)$$

$$IC_2(3) = C(1,3) - C(1)$$

$$IC_3(3) = C(1,2,3) - C(1,2)$$

$$IC_4(3) = C(3)$$

$$IC_5(3) = C(2,3) - C(2)$$

$$IC_6(3) = C(3)$$

Giving each of the possible orders equal weight, the average incremental cost for each polluter is obtained:

$$\rho(i) = \frac{1}{6} \sum_{j=1}^{6} IC_j(i) \quad , \quad i=1,2,3$$

Note that

$$\sum_{i=1}^{3} \rho(i) = C(1,2,3)$$

So that total costs would be covered by charging each polluter the amount $\rho(i)$.

Generalizing to the case of n polluters where there would be n! orderings, the cost allocation formula is given by

$$\rho(i) = \frac{1}{n!} \sum_{j=1}^{n!} IC_j(i) \quad , \quad i=1,\ldots,n \tag{30}$$

where $IC_j(i)$ is the incremental cost due to i in ordering j. Also,

$$\sum_{i=1}^{n} \rho(i) = C(1,2,\ldots,n)$$

This cost allocation scheme is discussed in [14]. Also in [15]

there is a theoretical development of the cost allocation method described. It is demonstrated that formula (30) is a unique formula which satisfies the following axioms:

1. Charges for the use of the basin treatment system must cover costs.

2. A user's charges are based only on the incremental cost caused by the user.

3. The user's charge is independent of the ordering or labeling of the users.

4. If a user's incremental costs increase by λ, then his charges will increase by λ.

The equity properties of this scheme are evident from the above axioms 2 - 4.

The following example illustrates cost allocation for the five-polluter case. In the report [5] the solution of a small scale pure by-pass piping problem is given on page 90. The dual solution for this problem is

x_1 = \$3285/MGD	x_9 = -\$1,929,780.MG/L
x_2 = \$11970/MGD	x_{10} = 0
x_3 = \$6445/MGD	x_{11} = 0
x_4 = \$2865/MGD	
x_5 = \$2381/MGD	

The extremal value can be calculated as follows

Q_1 = 19.7	Q_1x_1 = 64,714	K_1x_9 = -147,107
Q_2 = 7.0	Q_2x_2 = 83,790	
Q_3 = 3.0	Q_3x_3 = 19,335	
Q_4 = 6.1	Q_4x_4 = 17,476	194,839
Q_5 = 4.0	Q_5x_5 = 9,524	-147,107
K_1 = .07623	194,839	\$ 47,732

A pricing scheme suggested in [5] was to use the values of the x_iQ_i to give the proportion of the optimal solution to be paid by each polluter. This gives

Polluter	% Paid	Payment/Year
1	33.2	$15,643
2	43.0	20,261
3	9.9	4,664
4	8.9	4,146
5	4.8	2,261

It would be interesting to compare the costs from the global scheme to those given above. This is economically feasible since we only have five polluters. The coalition costs were obtained from the optimal solutions of 31 L.P. problems and are given as

C(1,2,3,4,5)	= $47,119	C(3,4,5)	= 0
C(1,2,3,5)	= 30,763	C(1,2,5)	= $25,238
C(1,2,4,5)	= 29,365	C(2,3,4)	= 16,690
C(1,3,4,5)	= 0	C(1,3,4)	= 0
C(2,3,4,5)	= 18,939	C(1,2,4)	= 27,115
C(1,2,3,4)	= 37,903	C(1,2,3)	= 27,844
C(1,3,5)	= 0	C(1,2)	= 22,988
C(1,4,5)	= 0	C(1,3)	= 0
C(2,3,5)	= 14,812	C(2,3)	= 12,563
C(2,4,5)	= 14,084	C(1,4)	= 0
C(2,4)	= 11,902	C(5)	= 0
C(3,4)	= 0	C(4)	= 0
C(1,5)	= 0	C(3)	= 0
C(2,5)	= $11,210	C(2)	= $10,381
C(3,5)	= 0	C(1)	= 0
C(4,5)	= 0		

The polluters not in the coalition were allowed to return their wastes to the estuary at no cost which in effect made them non-existent. The global scheme gave the following cost allocation:

Polluter	Local Method	Global Method
1	$15,643	$10,372
2	20,261	24,474
3	4,664	5,159
4	4,146	4,633
5	2,261	2,478

Application of Pricing Scheme

In the last subdivision we presented an alternate cost allocation mechanism to be used for regional treatment systems. In this subdivision we apply the scheme to be the Delaware Estuary Model.

We now wish to divide the total cost of the regional system, in this case the cost of the plants and pipes, between the polluters using the cost allocation method explained in the last subdivision. This necessitates the calculation of $C(G)$, $G \leq N$ where G ranges over possible subsets of polluters. The interpretation of $C(G)$ in terms of river basin model is not straightforward. For example, how do we compute $C(j)$ and what do we assume about polluters $I \neq j$. Are they non-existent, are they treating at some uniform level? Are they dumping their effluent with no treatment? An infinite number of assumptions could be made concerning the behavior of the polluters not in the coalition to be examined.

The only restraint on the behavior of polluters outside of the group being considered is that they are treating in a manner which will allow the polluters to gain feasibility of the basin problem (i.e., the quality constraints can be satisfied for some level of treatment of wastes). For example, if the polluters outside of the coalition are assumed to be dumping all wastes directly into the stream it is very likely that several sections of the river will violate the quality constraints no matter what the polluters do. One procedure which can guarantee the feasibility of the problem is to find the least cost solution to the basin problem, $C(N)$, and assume that all polluters outside of G are required to treat their effluent to the same level as they did in the solution for $C(N)$.

The term $C(N)$ can now be interpreted as the least-cost the polluters in G must pay to maintain the quality goals given that the other polluters in the basin are treating on site at the same level required for the least cost base solution. The polluters in G can make use of regional plants and piping. In the special case where G contains only one polluter piping is

the only alternative method available. In some problems no savings over on-site treatment may result for some coalitions of polluters. For example, if polluters in the coalition are so far apart that piping costs overwhelm any economies of scale gained from a regional plant, the best possible alternative for these polluters is to treat on site. In such a case, the cost to the coalition would just be the sum of the individual on-site costs, or

$$C(G) = \sum_{i \in G} C(i)$$

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27 Abstract

This report extends the work contained in the earlier work, "Mathematical Programming for Regional Water Quality Management." A mixed integer, continuous variable non-linear programming model is developed which promises to be much more realistic and effective in selecting an optimal configuration of regional treatment plants and pipe juncture nodes. A new Bender's type decomposition for the non-linear model is also presented as an appropriate and feasible solution technique. In the area of the physical response of the environment, the earlier work linked the dissolved oxygen profile and the control options through a constant transfer matrix obtained as a steady state solution to a system of differential equations. This work shows how the transfer coefficients can be recalculated periodically in the course of the system optimization. Full scale computations are carried out demonstrating this capability. The theoretical framework for including two additional treatment methods, mechanical re-aeration and flow augmentation via reservoirs is also developed.

Abstract	Institution