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Derivation of a Non-Boussinesq Set of Equations for an Atmospheric Shear Layer

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by

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1. INTRODUCTION

Volume I of this report (prepared for the Environmental Protection Agency in connection with EPA Contract No. 68-02-0014) presented a detailed discussion of the method of invariant modeling as it is applied to the problem of the computation of turbulence in an atmospheric shear layer and to the dispersal of pollutants in such a layer. In Volume I, the basic equations used are the familiar Boussinesq approximations. These equations are derived by considering the atmospheric motion to be a small departure from an adiabatic atmosphere at rest and further assuming that the extent of the shear layer being investigated is small compared to the scale of the atmosphere - this latter scale being of order $\rho_o/(\partial\rho_o/\partial z)$ where ρ_o is the density of the undisturbed adiabatic atmosphere and z is the altitude. Within the Boussinesq approximation, the divergence of the velocity field of the motion under study may be neglected. Since this divergence may not be considered zero for motions whose scale is of the order of the atmosphere, i.e., 7000 meters, we at A.R.A.P. have used equations for much of our work that are approximate equations, akin to the Boussinesq approximation, but which do not consider that the divergence of the velocity field is zero. This set of equations was used in Reference 1 and was also used in A.R.A.P. Report No. 169 [Ref. 2], A.R.A.P.'s first report to EPA on the application of invariant modeling to the dispersal of pollutants in the atmosphere.

In this report (Volume II of A.R.A.P. Report No. 186), the equations which represent a non-Boussinesq set of approximate equations for the motion of the atmosphere and which are used in References 1 and 2 are derived.

2. DERIVATION OF BASIC EQUATIONS

The equations which we shall take to govern the motion of a compressible, nonchemically-reacting perfect gas are given below. They are:

the perfect gas law

$$\tilde{p} = \tilde{\rho}\tilde{R}\tilde{T} \tag{2.1}$$

the continuity equation

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\rho}\tilde{u}_j) = 0 \tag{2.2}$$

the momentum equation

$$\frac{\partial \tilde{\rho}\tilde{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\rho}\tilde{u}_j\tilde{u}_i) = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\tilde{\tau}_{ij} - \tilde{\rho}g_i \tag{2.3}$$

where the stress due to molecular diffusion is given by

$$\tilde{\tau}_{ij} = \tilde{\mu}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) + \delta_{ij}\tilde{\mu}^* \frac{\partial \tilde{u}_m}{\partial x_m} \tag{2.4}$$

the energy equation

$$\frac{\partial \tilde{\rho}\tilde{h}}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\rho}\tilde{u}_j\tilde{h}) = \frac{\partial \tilde{p}}{\partial t} + \tilde{u}_j\frac{\partial \tilde{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\tilde{k}\frac{\partial \tilde{T}}{\partial x_j}\right) + \tilde{\tau}_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} \tag{2.5}$$

the species conservation equation

$$\frac{\partial}{\partial t}(\tilde{\rho}\tilde{C}_\alpha) + \frac{\partial}{\partial x_j}(\tilde{\rho}\tilde{u}_j\tilde{C}_\alpha) = \frac{\partial}{\partial x_j}\left(\tilde{\rho}\tilde{\mathcal{D}}\frac{\partial}{\partial x_j}\tilde{C}_\alpha\right) \tag{2.6}$$

In what follows, we will be interested in the development of turbulence and the transport of matter in the atmosphere.

We will assume that the matter to be transported in the atmosphere is not too highly concentrated, so that it makes no first-order effect upon the heat capacity or gas constant of the air in which it is carried. In this case, if we designate the heat capacity as C_{p_o} and the gas constant as R_o, we may write (2.1) as

$$\tilde{p} = \tilde{\rho} R_o \tilde{T} \tag{2.7}$$

and (2.5) as

$$C_{p_o}\left[\frac{\partial}{\partial t}\tilde{\rho}\tilde{T} + \frac{\partial}{\partial x_j}(\tilde{\rho}\tilde{u}_j\tilde{T})\right] = \frac{\partial \tilde{p}}{\partial t} + \tilde{u}_j\frac{\partial \tilde{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\tilde{k}\frac{\partial \tilde{T}}{\partial x_j} + \tilde{\tau}_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} \tag{2.8}$$

Since the Mach number of the flow in which we are interested is small, we may neglect the last term on the right-hand side of (2.8) since this term represents the heat generated by the dissipation of the motion and is of order M^2 compared with the other terms in the equation. The final form of the energy equation is, then,

$$C_{p_o}\left[\frac{\partial \tilde{\rho}\tilde{T}}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\rho}\tilde{u}_j\tilde{T})\right] = \frac{\partial \tilde{p}}{\partial t} + \tilde{u}_j\frac{\partial \tilde{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\tilde{k}\frac{\partial \tilde{T}}{\partial x_j}\right) \tag{2.9}$$

Following the usual practice for obtaining the equations for the motion of an atmospheric shear layer [Refs. 3 and 4], we will consider the atmosphere to be in a state slightly removed from an adiabatic atmosphere at rest. We consider then an expansion of the equations presented above according to the following scheme:

$$\begin{aligned} \tilde{p} &= p_o + p & \qquad \tilde{C}_\alpha &= C_{\alpha_o} + C_\alpha \\ \tilde{\rho} &= \rho_o + \rho & \tilde{\mu} &= \mu_o + \mu \\ \tilde{T} &= T_o + T & \tilde{\mu}^* &= \mu_o^* + \mu^* \\ \tilde{u}_j &= 0 + u_j & \tilde{k} &= k_o + k \\ \tilde{\rho}\tilde{\mathcal{O}} &= \rho_o\mathcal{O}_o + \rho\mathcal{O}_o + \rho_o\mathcal{O} + \rho\mathcal{O} \end{aligned} \tag{2.10}$$

If we expand the gas law (2.7) according to this scheme, we have

$$p_o = \rho_o R_o T_o \tag{2.11}$$

and

$$p = R_o(\rho_o T + \rho T_o + \rho T) \tag{2.12}$$

If we neglect the second-order term on the right-hand side of (2.12), we have

$$p = R_o(\rho_o T + \rho T_o) \tag{2.13}$$

The continuity equation (2.2) yields

$$\frac{\partial \rho_o}{\partial t} = 0 \tag{2.14}$$

which agrees with our assumption of a steady state, and

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho_o u_j + \rho u_j) = 0 \tag{2.15}$$

which, to first-order, can be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho_o u_j) = 0 \tag{2.16}$$

The momentum equation (2.3) yields

$$\frac{\partial p_o}{\partial x_i} = -\rho_o g_i \tag{2.17}$$

and

$$\frac{\partial}{\partial t}(\rho_o + \rho)u_i + \frac{\partial}{\partial x_j}\left[(\rho_o + \rho)u_i u_j\right] = -\frac{\partial p}{\partial x_i} - \rho g_i$$

$$+ \frac{\partial}{\partial x_j}\left[(\mu_o + \mu)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}(\mu_o^* + \mu^*)\frac{\partial u_m}{\partial x_m}\right] \tag{2.18}$$

To first order, this may be written

$$\frac{\partial}{\partial t}(\rho_o u_i) + \frac{\partial}{\partial x_j}(\rho_o u_i u_j) = -\frac{\partial p}{\partial x_i} - \rho g_i$$
$$+ \frac{\partial}{\partial x_j}\left[\mu_o\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\mu_o^* \frac{\partial u_m}{\partial x_m}\right] \qquad (2.19)$$

Expansion of the energy equation (2.9) yields

$$C_{p_o} \frac{\partial \rho_o T_o}{\partial t} = \frac{\partial p_o}{\partial t} + \frac{\partial}{\partial x_j}\left(k_o \frac{\partial T_o}{\partial x_j}\right) \qquad (2.20)$$

which, according to our assumption of steady base flow, requires

$$\frac{\partial}{\partial x_j}\left(k_o \frac{\partial T_o}{\partial x_j}\right) = 0 \qquad (2.21)$$

and

$$C_{p_o}\left\{\frac{\partial}{\partial t}(\rho T_o) + \frac{\partial}{\partial t}(\rho_o + \rho)T\right\}$$
$$+ \frac{\partial}{\partial x_j}\left[(\rho_o + \rho)u_j T_o + (\rho_o + \rho)u_j T\right]$$
$$= u_j \frac{\partial p_o}{\partial x_j} + \frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j}$$
$$+ \frac{\partial}{\partial x_j}\left[(k_o + k)\frac{\partial T}{\partial x_j} + k \frac{\partial T_o}{\partial x_j}\right] \qquad (2.22)$$

To first order, this may be written

$$C_{p_o}\left\{\frac{\partial}{\partial t}(\rho_o T) + \frac{\partial}{\partial x_j}(\rho_o u_j T)\right\} + C_{p_o} T_o\left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho_o u_j)\right\}$$
$$= u_j\left(\frac{\partial p_o}{\partial x_j} - \rho_o C_{p_o}\frac{\partial T_o}{\partial x_j}\right) + \frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j}$$
$$+ \frac{\partial}{\partial x_j}\left(k_o \frac{\partial T}{\partial x_j} + k \frac{\partial T_o}{\partial x_j}\right) \qquad (2.23)$$

In view of the continuity equation (2.16), we may neglect the second term on the left-hand side of (2.23). Also, because of our assumption that the undisturbed atmosphere is adiabatic, i.e.,

$$\frac{\partial p_o}{\partial x_j} = \rho_o C_{p_o} \frac{\partial T_o}{\partial x_j} \tag{2.24}$$

we may neglect the first term on the right-hand side of (2.23). Finally, the terms $\partial p/\partial t$ and $u_j\, \partial p/\partial x_j$ taken together represent the compressional heating due to the disturbed motion considered apart from gravitational effects, are of order M^2, and may be neglected. Therefore, (2.23) may be written

$$C_{p_o}\left\{\frac{\partial}{\partial t}(\rho_o T) + \frac{\partial}{\partial x_j}(\rho_o u_j T)\right\} = \frac{\partial}{\partial x_j}\left(k_o \frac{\partial T}{\partial x_j} + k \frac{\partial T_o}{\partial x_j}\right) \tag{2.25}$$

Substitution of the expansion (2.10) in the species conservation equation (2.6) yields

$$\frac{\partial \rho_o C_{\alpha_o}}{\partial t} = \frac{\partial}{\partial x_j}\left(\rho_o \mathcal{D}_o \frac{\partial C_{\alpha_o}}{\partial x_j}\right) \tag{2.26}$$

which is satisfied since we have assumed a steady base atmospheric condition in which C_{α_o} is a constant, and

$$\frac{\partial}{\partial t}(\rho_o + \rho)C_\alpha + \frac{\partial}{\partial x_j}(\rho_o + \rho)u_j C_\alpha$$

$$= \frac{\partial}{\partial x_j}\left[(\rho_o + \rho)\mathcal{D}_o \frac{\partial C_\alpha}{\partial x_j} + (\rho_o + \rho)\mathcal{D}\frac{\partial C_\alpha}{\partial x_j}\right] \tag{2.27}$$

To first order, this equation becomes

$$\frac{\partial \rho_o C_\alpha}{\partial t} + \frac{\partial}{\partial x_j}(\rho_o u_j C_\alpha) = \frac{\partial}{\partial x_j}\left(\rho_o \mathcal{D}_o \frac{\partial C_\alpha}{\partial x_j}\right) \tag{2.28}$$

We return now to the equations that result from the perfect gas law, namely, (2.11) and (2.13). If we divide (2.11) by (2.13), we obtain

$$\frac{p}{p_o} = \frac{T}{T_o} + \frac{\rho}{\rho_o} \tag{2.29}$$

Now we note that the changes in p will be of order $\rho_o u_j^2$, so that

$$\frac{T}{T_o} + \frac{\rho}{\rho_o} = O\left(\frac{u_j^2}{R_o T_o}\right) = O\left(\gamma M^2\right) \tag{2.30}$$

We will, therefore, neglect the effects of motion-induced pressure changes on the variation of density and temperature and take

$$\rho = -\frac{\rho_o T}{T_o} \tag{2.31}$$

This assumption is related to, and entirely consistent with, our neglect of the pressure work terms in the energy equation.

We may now derive an equation for the divergence of the velocity field. If (2.31) is substituted into the continuity equation (2.15), the result is, after some manipulation,

$$\frac{\partial \rho_o T}{\partial t} = T_o \frac{\partial}{\partial x_j} (\rho_o u_j) - T_o \frac{\partial}{\partial x_j}\left(\rho_o \frac{u_j T}{T_o}\right) \tag{2.32}$$

The energy equation (2.25) may be written

$$\frac{\partial \rho_o T}{\partial t} = -\frac{\partial}{\partial x_j} (\rho_o u_j T) + \frac{1}{C_{p_o}} \frac{\partial}{\partial x_j}\left(k_o \frac{\partial T}{\partial x_j} + k \frac{\partial T_o}{\partial x_j}\right) \tag{2.33}$$

Equating (2.32) and (2.33) results in

$$\frac{\partial}{\partial x_j} (\rho_o u_j) = -\rho_o u_j \frac{T}{T_o^2} \frac{\partial T_o}{\partial x_j} + \frac{1}{C_{p_o} T_o} \frac{\partial}{\partial x_j}\left(k_o \frac{\partial T}{\partial x_j} + k \frac{\partial T_o}{\partial x_j}\right) \tag{2.34}$$

or

$$\frac{\partial u_j}{\partial x_j} = -\frac{u_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} - \frac{T}{T_o} \frac{u_j}{T_o} \frac{\partial T_o}{\partial x_j} + \frac{R_o}{C_{p_o} p_o} \frac{\partial}{\partial x_j}\left(k_o \frac{\partial T}{\partial x_j} + k \frac{\partial T_o}{\partial x_j}\right) \tag{2.35}$$

Since $(1/\rho_o)(\partial\rho_o/\partial x_j)$ is of the same order as $(1/T_o)(\partial T_o/\partial x_j)$, that is, of the order of an inverse atmospheric scale L_a^{-1}, we may, by virtue of the T/T_o in the second term on the right-hand side of (2.35), neglect this term relative to the first. Resort to a high Reynolds number argument [Ref. 5] shows that we may also neglect the third term on the right-hand side of (2.35) in comparison with the first. Thus, the equation for the divergence of the velocity field may be written

$$\frac{\partial u_j}{\partial x_j} = - \frac{u_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \qquad (2.36)$$

This equation may now be used to replace the continuity equation (2.16).

3. DISCUSSION OF THE STEADY ATMOSPHERE

In the previous section, the equations that were obtained for the basic atmosphere, which we wish to be adiabatic and in a steady state, were:

the perfect gas law

$$p_o = \rho_o R_o T_o \tag{3.1}$$

continuity

$$\frac{\partial \rho_o}{\partial t} = 0 \tag{3.2}$$

momentum

$$\frac{\partial p_o}{\partial x_j} = - \rho_o g_j \tag{3.3}$$

energy

$$C_{p_o} \frac{\partial \rho_o T_o}{\partial t} - \frac{\partial p_o}{\partial t} = \frac{\partial}{\partial x_j} \left(k_o \frac{\partial T_o}{\partial x_j} \right) \tag{3.4}$$

or, in view of (3.1) and (3.2)

$$\rho_o C_{p_o} \frac{\partial T_o}{\partial t} = \gamma_o \frac{\partial}{\partial x_j} \left(k_o \frac{\partial T_o}{\partial x_j} \right) \tag{3.5}$$

where $\gamma_o = C_{p_o}/C_{v_o}$. In addition we had the adiabatic condition

$$\frac{\partial p_o}{\partial x_j} = \rho_o C_{p_o} \frac{\partial T_o}{\partial x_j} \tag{3.6}$$

Substitution of (3.3) into (3.6) establishes the adiabatic lapse rate

$$\frac{\partial T_o}{\partial x_j} = - \frac{g_j}{C_{p_o}} \tag{3.7}$$

or, if $g_j = \delta_{3j} g$ = constant and (x_1, x_2, x_3) is defined as (x,y,z)

$$\left.\begin{aligned} \frac{\partial T_o}{\partial x} &= \frac{\partial T_o}{\partial y} = 0 \\ \frac{\partial T}{\partial z} &= - \frac{g}{C_{p_o}} \end{aligned}\right. \tag{3.8}$$

Substitution of (3.8) into (3.5) yields

$$\rho_o C_{p_o} \frac{\partial T_o}{\partial t} = - \gamma_o \frac{\partial}{\partial z} \left(\frac{k_o g}{C_{p_o}} \right) \tag{3.9}$$

Since the thermal conductivity is a function of the temperature T_o , we see from (3.9) that the basic atmosphere we have assumed cannot be in true equilibrium. However, an order-of-magnitude analysis shows that the time constant for the rise in T_o with time is so long compared to most motions in which we are interested that for all practical purposes it is legitimate to consider $\partial T_o/\partial t = 0$.

4. DERIVATION OF THE EQUATIONS FOR A TURBULENT SHEAR LAYER

Collection of the equations for an atmospheric motion that were derived in Section 2 results in the following set:

Continuity

$$\frac{\partial u_j}{\partial x_j} = - \frac{u_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{4.1}$$

Momentum

$$\frac{\partial}{\partial t} (\rho_o u_i) + \frac{\partial}{\partial x_j} (\rho_o u_i u_j) = - \frac{\partial p}{\partial x_i} - g_i \rho + \frac{\partial}{\partial x_j} \left[\mu_o \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \mu_o^* \frac{\partial u_m}{\partial x_m} \right] \tag{4.2}$$

Energy

$$\frac{\partial}{\partial t} (\rho_o T) + \frac{\partial}{\partial x_j} (\rho_o u_j T) = \frac{\partial}{\partial x_j} \left(k_o \frac{\partial T}{\partial x_j} + k \frac{\partial T_o}{\partial x_j} \right) \tag{4.3}$$

Species Mass Fraction

$$\frac{\partial}{\partial t} (\rho_o C_\alpha) + \frac{\partial}{\partial x_j} (\rho_o u_j C_\alpha) = \frac{\partial}{\partial x_j} \left(\rho_o \mathcal{D}_o \frac{\partial C_\alpha}{\partial x_j} \right) \tag{4.4}$$

where ρ is given by

$$\rho = - \frac{\rho_o}{T_o} T \tag{4.5}$$

The set of equations used in A.R.A.P. Report No. 169 [Ref. 2] was derived from this set as follows.

First, it was assumed that the molecular transport coefficients μ_o, μ_o^*, k_o, and $\rho_o \mathcal{D}_o$ were constant.

Second, it was assumed that the Prandtl and Schmidt numbers, $\mu_o C_{p_o}/k_o$ and $\mu_o/\rho_o \mathscr{D}_o$, of the medium were close enough to one so that we could write

$$k_o = \mu_o C_{p_o} \quad \text{and} \quad k = 0 \tag{4.6}$$

and

$$\rho_o \mathscr{D}_o = \mu_o \tag{4.7}$$

Equations (4.2), (4.3), and (4.4) then simplify to

$$\frac{\partial}{\partial t}(\rho_o u_i) + \frac{\partial}{\partial x_j}(\rho_o u_i u_j) = -\frac{\partial p}{\partial x_i} - g_i \rho + \mu_o \frac{\partial^2 u_i}{\partial x_j^2} + (\mu_o + \mu_o^*) \frac{\partial}{\partial x_i} \frac{\partial u_m}{\partial x_m} \tag{4.8}$$

$$\frac{\partial}{\partial t}(\rho_o T) + \frac{\partial}{\partial x_j}(\rho_o u_j T) = \mu_o \frac{\partial^2 T}{\partial x_j^2} \tag{4.9}$$

$$\frac{\partial}{\partial t}(\rho_o C_\alpha) + \frac{\partial}{\partial x_j}(\rho_o u_j C_\alpha) = \mu_o \frac{\partial^2 C_\alpha}{\partial x_j^2} \tag{4.10}$$

We now assume the velocity, pressure, temperature, density, and species mass fraction fields to be composed of mean and fluctuating parts according to the following scheme:

$$\begin{aligned} u_i &= \bar{u}_i + u_i' \\ p &= \bar{p} + p' \\ T &= \bar{T} + T' \\ \rho &= \bar{\rho} + \rho' \\ C_\alpha &= \bar{C}_\alpha + C_\alpha' \end{aligned} \tag{4.11}$$

Substitution of this scheme into (4.1) yields

$$\frac{\partial \bar{u}_j}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} = - \frac{\bar{u}_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} - \frac{u'_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{4.12}$$

Averaging this equations yields

$$\frac{\partial \bar{u}_j}{\partial x_j} = - \frac{\bar{u}_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{4.13}$$

or, alternatively,

$$\frac{\partial}{\partial x_j} (\rho_o \bar{u}_j) = 0 \tag{4.14}$$

Subtracting (4.13) from (4.12) gives

$$\frac{\partial u'_j}{\partial x_j} = - \frac{u'_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{4.15}$$

or, alternatively,

$$\frac{\partial}{\partial x_j} (\rho_o u'_j) = 0 \tag{4.16}$$

Substituting the scheme (4.11) in (4.5) results in

$$\bar{\rho} = - \frac{\rho_o}{T_o} \bar{T} \tag{4.17}$$

and

$$\rho' = - \frac{\rho_o}{T_o} T' \tag{4.18}$$

When (4.11) is substituted in (4.8), the result is

$$\frac{\partial}{\partial t} (\rho_o \bar{u}_i) + \frac{\partial}{\partial t} (\rho_o u'_i) + \frac{\partial}{\partial x_j} \left(\rho_o \bar{u}_i \bar{u}_j + \rho_o \bar{u}_i u'_j + \rho_o u'_i \bar{u}_j + \rho_o u'_i u'_j \right)$$

$$= - \frac{\partial \bar{p}}{\partial x_i} - \frac{\partial p'}{\partial x_i} - \bar{\rho} g_i - \rho' g_i + \mu_o \frac{\partial^2 \bar{u}_i}{\partial x_j^2} + \mu_o \frac{\partial^2 u'_i}{\partial x_j^2}$$

$$+ (\mu_o + \mu_o^*) \frac{\partial}{\partial x_i} \left(\frac{\partial \bar{u}_m}{\partial x_m} + \frac{\partial u'_m}{\partial x_m} \right) \tag{4.19}$$

Averaging this result yields

$$\frac{\partial \rho_o \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho_o \bar{u}_i \bar{u}_j)$$

$$= -\frac{\partial \bar{p}}{\partial x_i} - \bar{\rho} g_i + \mu_o \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \frac{\partial}{\partial x_j}(\rho_o \overline{u_i' u_j'})$$

$$+ (\mu_o + \mu_o^*) \frac{\partial}{\partial x_i}\left(\frac{\partial \bar{u}_m}{\partial x_m}\right) \tag{4.20}$$

Subtracting (4.20) from (4.19) yields an equation for the fluctuating velocity u_i' , namely,

$$\frac{\partial}{\partial t}(\rho_o u_i') + \frac{\partial}{\partial x_j}\left(\rho_o \bar{u}_i u_j' + \rho_o u_i' \bar{u}_j + \rho_o u_i' u_j' - \rho_o \overline{u_i' u_j'}\right)$$

$$= -\frac{\partial p'}{\partial x_j} - \rho' g_i + \mu_o \frac{\partial^2 u_i'}{\partial x_j^2} + (\mu_o + \mu_o^*) \frac{\partial}{\partial x_i}\left(\frac{\partial u_m'}{\partial x_m}\right) \tag{4.21}$$

If this equation is multiplied by u_k' and the resulting equation written again with the indices i and k reversed and then the two equations added together and averaged, the result is

$$\frac{\partial}{\partial t}(\rho_o \overline{u_i' u_k'}) + \frac{\partial}{\partial x_j}(\rho_o \bar{u}_j \overline{u_i' u_k'})$$

$$= -\rho_o \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j' u_i'} \frac{\partial \bar{u}_k}{\partial x_j} - \frac{\partial}{\partial x_j}(\rho_o \overline{u_i' u_j' u_k'})$$

$$- \frac{\partial}{\partial x_i}(\overline{p' u_k'}) - \frac{\partial}{\partial x_k}(\overline{p' u_i'}) + \overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)}$$

$$- \overline{\rho' u_k'} g_i - \overline{\rho' u_i'} g_k + \mu_o \left(\overline{u_k' \frac{\partial^2 u_i'}{\partial x_j^2}} + \overline{u_i' \frac{\partial^2 u_k'}{\partial x_j^2}}\right)$$

$$+ (\mu_o + \mu_o^*) \overline{\left(u_i' \frac{\partial}{\partial x_k} + u_k' \frac{\partial}{\partial x_i}\right)\left(\frac{\partial u_m'}{\partial x_m}\right)} \tag{4.22}$$

We may modify (4.20) by using (4.13) and (4.17) and write

$$\frac{\partial \rho_o \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho_o \bar{u}_i \bar{u}_j) = - \frac{\partial \bar{p}}{\partial x_i} + \frac{\rho_o}{T_o} \bar{T} g_i + \mu_o \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\rho_o \overline{u_i' u_j'}) - (\mu_o + \mu_o^*) \frac{\partial}{\partial x_i} \left(\frac{\bar{u}_m}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) \tag{4.23}$$

In the same way, we may modify (4.22) through use of (4.15) and (4.18) to obtain, after noting that

$$u_k' \frac{\partial^2 u_i'}{\partial x_j^2} + u_i' \frac{\partial^2 u_k'}{\partial x_j^2} = \frac{\partial^2 \overline{u_i' u_k'}}{\partial x_j^2} - 2 \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} \tag{4.24}$$

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_o \overline{u_i' u_k'}) &+ \frac{\partial}{\partial x_j} (\rho_o \bar{u}_j \overline{u_i' u_k'}) \\ &= - \rho_o \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j' u_i'} \frac{\partial \bar{u}_k}{\partial x_j} - \frac{\partial}{\partial x_j} (\rho_o \overline{u_i' u_j' u_k'}) \\ &- \frac{\partial}{\partial x_i} (\overline{p' u_k'}) - \frac{\partial}{\partial x_k} (\overline{p' u_i'}) + \overline{p' \left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i} \right)} \\ &+ \frac{\rho_o}{T_o} \overline{T' u_k'} g_i + \frac{\rho_o}{T_o} \overline{T' u_i'} g_k \\ &+ \mu_o \frac{\partial^2 \overline{u_i' u_k'}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} \\ &- (\mu_o + \mu_o^*) \overline{\left(u_i' \frac{\partial}{\partial x_k} + u_k' \frac{\partial}{\partial x_i} \right) \left(\frac{u_m'}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right)} \end{aligned} \tag{4.25}$$

The momentum and Reynolds stress equations, (4.23) and (4.25), given above are the same as Eqs. (30) and (34) of Reference 2.

We now derive equations for the mean temperature field and the variance of the temperature fluctuations from (4.9). Substitution of the scheme (4.11) into (4.9) yields

$$\frac{\partial}{\partial t}(\rho_o \overline{T}) + \frac{\partial}{\partial t}(\rho_o T') + \frac{\partial}{\partial x_j}\left(\rho_o \bar{u}_j \overline{T} + \rho_o \bar{u}_j T' + \rho_o u_j' \overline{T} + \rho_o u_j' T'\right)$$

$$= \mu_o \frac{\partial^2 \overline{T}}{\partial x_j^2} + \mu_o \frac{\partial^2 T'}{\partial x_j^2} \tag{4.26}$$

Averaging this equation gives

$$\frac{\partial}{\partial t}(\rho_o \overline{T}) + \frac{\partial}{\partial x_j}(\rho_o \bar{u}_j \overline{T}) = \mu_o \frac{\partial^2 \overline{T}}{\partial x_j^2} - \frac{\partial}{\partial x_j}(\rho_o \overline{u_j' T'}) \tag{4.27}$$

If we subtract (4.27) from (4.26), we obtain an equation for the temperature fluctuation, namely,

$$\frac{\partial}{\partial t}(\rho_o T') + \frac{\partial}{\partial x_j}\left(\rho_o \bar{u}_j T' + \rho_o u_j' \overline{T} + \rho_o u_j' T' - \rho_o \overline{u_j' T'}\right) = \mu_o \frac{\partial^2 T'}{\partial x_j^2} \tag{4.28}$$

To obtain an equation for $\overline{T'^2}$, (4.28) may be multiplied by $2T'$ and the resulting equation averaged, with the result that

$$\frac{\partial}{\partial t}\left(\rho_o \overline{T'^2}\right) + \frac{\partial}{\partial x_j}\left(\rho_o \bar{u}_j \overline{T'^2}\right) = -2\rho_o \overline{T' u_j'} \frac{\partial \overline{T}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\rho_o \overline{u_j' T'^2}\right)$$

$$+ \mu_o \frac{\partial^2 \overline{T'^2}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j}} \tag{4.29}$$

The mean temperature equation and the variance equation, (4.27) and (4.29) are the same as Eqs. (31) and (36) of Reference 2.

An equation for the heat transport correlation $\overline{u_i'T'}$ can be obtained from (4.21) and (4.28). First, multiply (4.21) by T' and then multiply (4.28) by u_i'. The resulting two equations are then added and averaged; the result is

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho_o \overline{u_i'T'}) + \frac{\partial}{\partial x_j}(\rho_o \bar{u}_j \overline{u_i'T'}) \\
&\quad = -\rho_o \overline{u_j'T'}\frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j'u_i'}\frac{\partial \bar{T}}{\partial x_j} - \frac{\partial}{\partial x_j}(\rho_o \overline{u_i'u_j'T'}) \\
&\quad - \frac{\partial}{\partial x_j}(\overline{T'p'}) + \overline{p'\frac{\partial T'}{\partial x_j}} + \frac{\rho_o}{T_o}\overline{T'^2}\, g_i \\
&\quad + \mu_o \frac{\partial^2 \overline{u_i'T'}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial T'}{\partial x_j}} \\
&\quad - (\mu_o + \mu_o^*)\overline{T'\frac{\partial}{\partial x_i}\left(\frac{u_m'}{\rho_o}\frac{\partial \rho_o}{\partial x_m}\right)}
\end{aligned} \tag{4.30}$$

This equation is identical to Eq. (35) of Reference 2.

To summarize the relationship between the basic equations given in Reference 2 and the equations derived here, one may refer to Table 4.1 below:

Table 4.1

Equation Number Reference 2	Equation Number This Report
30	4.23
31	4.27
32	4.13
33	4.15
34	4.25
35	4.30
36	4.29

We now derive a set of equations consistent with those just presented for the mean species mass fraction $\bar{C}_\alpha$, for the transport correlation $\overline{u_i'C_\alpha'}$, and for the correlations $\overline{C_\alpha'T'}$ and $\overline{C_\alpha'^2}$.

If the scheme (4.11) is substituted in (4.10), one obtains

$$\frac{\partial}{\partial t}(\rho_o\bar{C}_\alpha) + \frac{\partial}{\partial t}(\rho_o C_\alpha') + \frac{\partial}{\partial x_j}\left(\rho_o\bar{u}_j\bar{C}_\alpha + \rho_o\bar{u}_j C_\alpha' + \rho_o u_j'\bar{C}_\alpha + \rho_o u_j' C_\alpha'\right)$$
$$= \mu_o \frac{\partial^2\bar{C}_\alpha}{\partial x_j^2} + \mu_o \frac{\partial^2 C_\alpha'}{\partial x_j^2} \qquad (4.31)$$

Averaging this equation yields

$$\frac{\partial\rho_o\bar{C}_\alpha}{\partial t} + \frac{\partial}{\partial x_j}(\rho_o\bar{u}_j\bar{C}_\alpha) = \mu_o\frac{\partial^2\bar{C}_\alpha}{\partial x_j^2} - \frac{\partial}{\partial x_j}(\rho_o\overline{u_j'C_\alpha'}) \qquad (4.32)$$

Subtracting this expression from (4.31) yields an equation for C_α' , namely,

$$\frac{\partial}{\partial t}(\rho_o C_\alpha') + \frac{\partial}{\partial x_j}\left(\rho_o\bar{u}_j C_\alpha' + \rho_o u_j'\bar{C}_\alpha + \rho_o u_j' C_\alpha' - \rho_o\overline{u_j'C_\alpha'}\right)$$
$$= \mu_o\frac{\partial^2 C_\alpha'}{\partial x_j^2} \qquad (4.33)$$

Multiplying this equation by $2C_\alpha'$ and averaging yields

$$\frac{\partial}{\partial t}\left(\rho_o\overline{C_\alpha'^2}\right) + \frac{\partial}{\partial x_j}\left(\rho_o\bar{u}_j\overline{C_\alpha'^2}\right) = -2\rho_o\overline{u_j'C_\alpha'}\frac{\partial\bar{C}_\alpha}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\rho_o\overline{u_j'C_\alpha'^2}\right)$$
$$+ \mu_o\frac{\partial^2\overline{C_\alpha'^2}}{\partial x_j^2} - 2\mu_o\overline{\frac{\partial C_\alpha'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}} \qquad (4.34)$$

To obtain an equation for $\overline{u_i'C_\alpha'}$, we multiply (4.33) by u_i' and (4.21) by C_α' and add the two equations. If the resulting equation is averaged, one obtains

$$\frac{\partial}{\partial t}(\rho_o\overline{u_i'C_\alpha'}) + \frac{\partial}{\partial x_j}(\rho_o\bar{u}_j\overline{u_i'C_\alpha'})$$

$$= -\rho_o\overline{u_j'C_\alpha'}\frac{\partial\bar{u}_i}{\partial x_j} - \rho_o\overline{u_j'u_i'}\frac{\partial\bar{C}_\alpha}{\partial x_j} - \frac{\partial}{\partial x_j}(\rho_o\overline{u_i'u_j'C_\alpha'})$$

$$- \frac{\partial}{\partial x_i}(\overline{C_\alpha'p'}) + \overline{p'\frac{\partial C_\alpha'}{\partial x_i}} + \frac{\rho_o}{T_o}\overline{T'C_\alpha'}\,g_i$$

$$+ \mu_o\frac{\partial^2\overline{u_i'C_\alpha'}}{\partial x_j^2} - 2\mu_o\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}}$$

$$+ (\mu_o + \mu_o^*)\overline{C_\alpha'\frac{\partial}{\partial x_i}\left(\frac{\partial u_m'}{\partial x_m}\right)} \qquad (4.35)$$

The equation for $\overline{T'C_\alpha'}$ is obtained by multiplying (4.33) by T' and (4.28) by C_α' and adding the two equations. The resulting equation, when averaged, yields

$$\frac{\partial}{\partial t}(\rho_o\overline{T'C_\alpha'}) + \frac{\partial}{\partial x_j}(\rho_o\bar{u}_j\overline{T'C_\alpha'})$$

$$= -\rho_o\overline{u_j'T'}\frac{\partial\bar{C}_\alpha}{\partial x_j} - \rho_o\overline{u_j'C_\alpha'}\frac{\partial\bar{T}}{\partial x_j} - \frac{\partial}{\partial x_j}(\rho_o\overline{u_j'C_\alpha'T'})$$

$$+ \mu_o\frac{\partial^2\overline{T'C_\alpha'}}{\partial x_j^2} - 2\mu_o\overline{\frac{\partial T'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}} \qquad (4.36)$$

Equations (4.32), (4.34), (4.35), and (4.36) are the equations necessary to describe the dispersal of a passive pollutant in an atmospheric shear layer governed by the equations given in Reference 2. These equations are the basic equations from which those discussed in A.R.A.P. Report No. 169 were derived by invariant modeling.

We will now turn to a discussion of the modeling of the terms in the equations derived above that must be modeled to form a closed set of equations.

5. MODELING OF TERMS IN BASIC EQUATIONS TO ACHIEVE CLOSURE

First let us list the basic equations derived in the previous section which we will need to enable us to calculate the generation of turbulence in an atmospheric shear layer and the dispersal of a passive pollutant in this layer. In rewriting these equations, the forms of some will be slightly changed by use of the two continuity equations [(4.13) and (4.15)] and the condition that $\partial\rho_o/\partial t = 0$.

$$\rho_o \frac{\partial \bar{u}_i}{\partial t} + \rho_o \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = - \frac{\partial \bar{p}}{\partial x_i} + \frac{\rho_o \bar{T}}{T_o} g_i + \mu_o \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - (\rho_o \overline{u_i' u_j'}) - (\mu_o + \mu_o^*) \frac{\partial}{\partial x_i}\left(\frac{\bar{u}_m}{\rho_o}\frac{\partial \rho_o}{\partial x_m}\right) \tag{5.1}$$

$$\rho_o \frac{\partial \bar{T}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \bar{T}}{\partial x_j} = \mu_o \frac{\partial^2 \bar{T}}{\partial x_j^2} - \frac{\partial}{\partial x_j}(\rho_o \overline{u_j' T'}) \tag{5.2}$$

$$\frac{\partial \bar{u}_j}{\partial x_j} = - \frac{\bar{u}_j}{\rho_o}\frac{\partial \rho_o}{\partial x_j} \tag{5.3}$$

$$\frac{\partial u_j'}{\partial x_j} = - \frac{u_j'}{\rho_o}\frac{\partial \rho_o}{\partial x_j} \tag{5.4}$$

$$\rho_o \frac{\partial \overline{u_i' u_k'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{u_i' u_k'}}{\partial x_j} = - \rho_o \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j' u_i'} \frac{\partial \bar{u}_k}{\partial x_j}$$

$$- \frac{\partial}{\partial x_j} (\rho_o \overline{u_i' u_j' u_k'}) - \frac{\partial}{\partial x_i} (\overline{p' u_k'}) - \frac{\partial}{\partial x_k} (\overline{p' u_i'})$$

$$+ \overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} + \frac{\rho_o}{T_o} \overline{T' u_k'} g_i + \frac{\rho_o}{T_o} \overline{T' u_i'} g_k$$

$$+ \mu_o \frac{\partial^2 \overline{u_i' u_k'}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}}$$

$$- (\mu_o + \mu_o^*) \left[\overline{u_i' u_m'} \frac{\partial}{\partial x_k} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) + \overline{u_k' u_m'} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) \right.$$

$$\left. + \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \left(\overline{u_i' \frac{\partial u_m'}{\partial x_k}} + \overline{u_k' \frac{\partial u_m'}{\partial x_i}} \right) \right] \qquad (5.5)$$

$$\rho_o \frac{\partial \overline{u_i' T'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{u_i' T'}}{\partial x_j} = - \rho_o \overline{u_j' T'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j' u_i'} \frac{\partial \bar{T}}{\partial x_j}$$

$$- \frac{\partial}{\partial x_j} (\rho_o \overline{u_i' u_j' T'}) - \frac{\partial}{\partial x_i} (\overline{T' p'}) + \overline{p' \frac{\partial T'}{\partial x_i}} + \frac{\rho_o}{T_o} \overline{T'^2} g_i$$

$$+ \mu_o \frac{\partial^2 \overline{u_i' T'}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial T'}{\partial x_j}}$$

$$- (\mu_o + \mu_o^*) \left[\overline{T' u_m'} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) + \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \overline{T' \frac{\partial u_m'}{\partial x_i}} \right]$$

$$(5.6)$$

$$\rho_o \frac{\partial \overline{T'^2}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{T'^2}}{\partial x_j} = - 2\rho_o \overline{T' u_j'} \frac{\partial \bar{T}}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\rho_o \overline{u_j' T'^2} \right)$$

$$+ \mu_o \frac{\partial^2 \overline{T'^2}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j}} \qquad (5.7)$$

$$\rho_o \frac{\partial \bar{C}_\alpha}{\partial t} + \rho_o \bar{u}_j \frac{\partial \bar{C}_\alpha}{\partial x_j} = \mu_o \frac{\partial^2 \bar{C}_\alpha}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\rho_o \overline{u_j' C_\alpha'}) \tag{5.8}$$

$$\begin{aligned}\rho_o \frac{\partial \overline{u_i' C_\alpha'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{u_i' C_\alpha'}}{\partial x_j} &= - \rho_o \overline{u_j' C_\alpha'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j' u_i'} \frac{\partial \bar{C}_\alpha}{\partial x_j} \\ &- \frac{\partial}{\partial x_j} (\rho_o \overline{u_i' u_j' C_\alpha'}) - \frac{\partial}{\partial x_i} (\overline{C_\alpha' p'}) + \overline{p' \frac{\partial C_\alpha'}{\partial x_i}} + \frac{\rho_o}{T_o} \overline{T' C_\alpha'} g_i \\ &+ \mu_o \frac{\partial^2 \overline{u_i' C_\alpha'}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial C_\alpha'}{\partial x_j}} \\ &- (\mu_o + \mu_o^*) \left[\overline{C_\alpha' u_m'} \frac{\partial}{\partial x_i} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) + \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \overline{C_\alpha' \frac{\partial u_m'}{\partial x_i}} \right]\end{aligned} \tag{5.9}$$

$$\begin{aligned}\rho_o \frac{\partial \overline{C_\alpha' T'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{C_\alpha' T'}}{\partial x_j} &= - \rho_o \overline{u_j' T'} \frac{\partial \bar{C}_\alpha}{\partial x_j} - \rho_o \overline{u_j' C_\alpha'} \frac{\partial \bar{T}}{\partial x_j} \\ &- \frac{\partial}{\partial x_j} (\rho_o \overline{u_j' C_\alpha' T'}) + \mu_o \frac{\partial^2 \overline{T' C_\alpha'}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial T'}{\partial x_j} \frac{\partial C_\alpha'}{\partial x_j}}\end{aligned} \tag{5.10}$$

$$\begin{aligned}\rho_o \frac{\partial \overline{C_\alpha'^2}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{C_\alpha'^2}}{\partial x_j} &= - 2\rho_o \overline{u_j' C_\alpha'} \frac{\partial \bar{C}_\alpha}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\rho_o \overline{u_j' C_\alpha'^2} \right) \\ &+ \mu_o \frac{\partial^2 \overline{C_\alpha'^2}}{\partial x_j^2} - 2\mu_o \overline{\frac{\partial C_\alpha'}{\partial x_j} \frac{\partial C_\alpha'}{\partial x_j}}\end{aligned} \tag{5.11}$$

and, finally,

$$\bar{\rho} = - \frac{\rho_o}{T_o} \bar{T} \tag{5.12}$$

and

$$\rho' = - \frac{\rho_o}{T_o} T' \tag{5.13}$$

It will be noted that Eqs. (5.1) through (5.7) do not depend on the passive scalar C_α. Thus one can, in principle, solve for the turbulence in an atmospheric shear layer using (5.1) through (5.7) and with this information in hand solve for the dispersal of the pollutant field given by $\overline{C}_\alpha$. To do this, (5.8) through (5.10), which are independent of (5.11), may be solved for $\overline{C}_\alpha$. Finally, (5.11) may be solved for the variance $\overline{C_\alpha'^2}$. The mean density field and the variance of the density fluctuations may be obtained from (5.12) and (5.13) from the previous solutions for $\overline{T}$ and $\overline{T'^2}$.

What is needed to perform these calculations is a closure of the correlation equations given above. We give next a list of the terms that must be modeled, ordered according to the type of term.

a. Tendency-to-isotropy terms

In the equation for $\overline{u_i'u_k'}$, (5.5)

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)}$$

In the equation for $\overline{u_i'T'}$, (5.6)

$$\overline{p'\frac{\partial T'}{\partial x_i}}$$

In the equation for $\overline{u_i'C_\alpha'}$, (5.9)

$$\overline{p'\frac{\partial C_\alpha'}{\partial x_i}}$$

b. Velocity diffusion terms

In the equation for $\overline{u_i'u_k'}$, (5.5)

$$\rho_o\overline{u_i'u_j'u_k'}$$

In the equation for $\overline{u_i'T'}$, (5.6)

$$\rho_o\overline{u_i'u_j'T'}$$

In the equation for $\overline{T'^2}$, (5.7)

$$\rho_o \overline{u_j' T'^2}$$

In the equation for $\overline{u_i' C_\alpha'}$, (5.9)

$$\rho_o \overline{u_i' u_j' C_\alpha'}$$

In the equation for $\overline{T' C_\alpha'}$, (5.10)

$$\rho_o \overline{u_j' T' C_\alpha'}$$

In the equation for $\overline{C_\alpha'^2}$, (5.11)

$$\rho_o \overline{u_j' C_\alpha'^2}$$

c. <u>Pressure diffusion terms</u>

In the equation for $\overline{u_i' u_k'}$, (5.5)

$$\overline{p' u_i'} \quad \text{and} \quad \overline{p' u_k'}$$

In the equation for $\overline{u_i' T'}$, (5.6)

$$\overline{p' T'}$$

In the equation for $\overline{u_i' C_\alpha'}$, (5.9)

$$\overline{p' C_\alpha'}$$

d. <u>Dissipation terms</u>

In the equation for $\overline{u_i' u_k'}$, (5.5)

$$\overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}}$$

In the equation for $\overline{u_i' T'}$, (5.6)

$$\overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial T'}{\partial x_j}}$$

In the equation for $\overline{T'^2}$, (5.7)

$$\overline{\frac{\partial T'}{\partial x_j}\frac{\partial T'}{\partial x_j}}$$

In the equation for $\overline{u_i'C_\alpha'}$, (5.9)

$$\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}}$$

In the equation for $\overline{T'C_\alpha'}$, (5.10)

$$\overline{\frac{\partial T'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}}$$

In the equation for $\overline{C_\alpha'^2}$, (5.11)

$$\overline{\frac{\partial C_\alpha'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}}$$

e. <u>Divergence/strain terms</u>

In the equation for $\overline{u_i'u_k'}$, (5.5)

$$\overline{u_i' \frac{\partial u_m'}{\partial x_k}} \quad \text{and} \quad \overline{u_k' \frac{\partial u_m'}{\partial x_i}}$$

In the equation for $\overline{u_i'T'}$, (5.6)

$$\overline{T' \frac{\partial u_m'}{\partial x_i}}$$

In the equation for $\overline{u_i'C_\alpha'}$, (5.9)

$$\overline{C_\alpha' \frac{\partial u_m'}{\partial x_i}}$$

The modeling we will adopt for these terms follows the detailed discussion of modeling presented in Ref. 5. We will start with the tendency-towards-isotropy terms.

The model used in Ref. 5 in the $\overline{u_i'u_k'}$ equation was

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} = -\frac{\rho_o q}{\Lambda_1}\left(\overline{u_i'u_k'} - \delta_{ik}\frac{q^2}{3}\right)$$

This equation satisfies the condition for incompressible flows that when $i = k$ the model vanishes as it must, since the basic expression vanishes as a result of the velocity fields being nondivergent. In the case considered here, the divergence of the turbulent velocity field does not vanish, but is given by (5.4), namely,

$$\frac{\partial u_j'}{\partial x_j} = - \frac{u_j'}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{5.4}$$

We may change the model so as to agree with this result in two simple ways. We may write

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} = - \frac{\rho_o q}{\Lambda_1}\left(\overline{u_i' u_k'} - \delta_{ik}\frac{q^2}{3}\right) - 2\delta_{ik}\frac{\overline{p' u_m'}}{\rho_o}\frac{\partial \rho_o}{\partial x_m} \tag{5.14}$$

or

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} = - \frac{\rho_o q}{\Lambda_1}\left(\overline{u_i' u_k'} - \delta_{ik}\frac{q^2}{3}\right) - \frac{\overline{p' u_k'}}{\rho_o}\frac{\partial \rho_o}{\partial x_i} - \frac{\overline{p' u_i'}}{\rho_o}\frac{\partial \rho_o}{\partial x_k} \tag{5.15}$$

The first model above allows the divergence of the turbulence field to contribute only to turbulent energy directly; while the second model allows contribution of the divergence only to terms in an actual atmosphere which contains the vertical fluctuation since only $\partial \rho_o / \partial z$ exists. The second expression (5.15) is the modeling given in Ref. 2. After considerable thought, it is now felt that (5.14) might have been a more appropriate choice since a simple divergence of a fluid element will affect all velocities equally.

Following the model for the tendency towards isotropy in the $\overline{u_i' u_k'}$ equation, we choose

$$\overline{p' \frac{\partial T'}{\partial x_i}} = - \frac{\rho_o q}{\Lambda_1} \overline{u_i' T'} \tag{5.16}$$

and

$$\overline{p' \frac{\partial C_\alpha'}{\partial x_i}} = - \frac{\rho_o q}{\Lambda_1} \overline{u_i' C_\alpha'} \tag{5.17}$$

The models for the velocity diffusion terms can be taken over directly from our previous work. We choose

$$\rho_o \overline{u_i' u_j' u_k'} = - \rho_o \Lambda_2 q \left(\frac{\partial \overline{u_i' u_j'}}{\partial x_k} + \frac{\partial \overline{u_i' u_k'}}{\partial x_j} + \frac{\partial \overline{u_j' u_k'}}{\partial x_i} \right) \tag{5.18}$$

$$\rho_o \overline{u_i' u_j' T'} = - \rho_o \Lambda_2 q \left(\frac{\partial \overline{u_i' T'}}{\partial x_j} + \frac{\partial \overline{u_j' T'}}{\partial x_i} \right) \tag{5.19}$$

$$\rho_o \overline{u_i' u_j' C_\alpha'} = - \rho_o \Lambda_2 q \left(\frac{\partial \overline{u_i' C_\alpha'}}{\partial x_j} + \frac{\partial \overline{u_j' C_\alpha'}}{\partial x_i} \right) \tag{5.20}$$

$$\rho_o \overline{u_j' T'^2} = - \rho_o \Lambda_2 q \frac{\partial \overline{T'^2}}{\partial x_j} \tag{5.21}$$

$$\rho_o \overline{u_j' C_\alpha' T'} = - \rho_o \Lambda_2 q \frac{\partial \overline{C_\alpha' T'}}{\partial x_j} \tag{5.22}$$

$$\rho_o \overline{u_j' C_\alpha'^2} = - \rho_o \Lambda_2 q \frac{\partial \overline{C_\alpha'^2}}{\partial x_j} \tag{5.23}$$

The pressure diffusion terms may also be taken over from our previous work. We write, therefore,

$$\overline{p' u_i'} = - \rho_o \Lambda_3 q \frac{\partial \overline{u_i' u_j'}}{\partial x_j} \tag{5.24}$$

$$\overline{p'T'} = -\rho_o \Lambda_3 q \frac{\partial \overline{T'u_j'}}{\partial x_j} \tag{5.25}$$

$$\overline{p'C_\alpha'} = -\rho_o \Lambda_3 q \frac{\partial \overline{C_\alpha' u_j'}}{\partial x_j} \tag{5.26}$$

If we follow our previous work on incompressible turbulent shear flows, the modeling of the dissipation terms need not be changed. We have, therefore,

$$\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}} = \frac{\overline{u_i'u_k'}}{\lambda^2} \tag{5.27}$$

$$\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial T'}{\partial x_j}} = \frac{\overline{u_i'T'}}{\lambda^2} \tag{5.28}$$

$$\overline{\frac{\partial T'}{\partial x_j}\frac{\partial T'}{\partial x_j}} = \frac{\overline{T'^2}}{\lambda^2} \tag{5.29}$$

$$\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}} = \frac{\overline{u_i'C_\alpha'}}{\lambda^2} \tag{5.30}$$

$$\overline{\frac{\partial T'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}} = \frac{\overline{T'C_\alpha'}}{\lambda^2} \tag{5.31}$$

$$\overline{\frac{\partial C_\alpha'}{\partial x_j}\frac{\partial C_\alpha'}{\partial x_j}} = \frac{\overline{C_\alpha'^2}}{\lambda^2} \tag{5.32}$$

The divergence/strain terms did not appear at all in the case of an incompressible shear layer. They are new and must be modeled for the first time. Consider the term $\overline{u_i'(\partial u_m'/\partial x_k)}$.

There are two simple ways in which it might be modeled.

$$\overline{u_i' \frac{\partial u_m'}{\partial x_k}} = - \delta_{mk} \frac{\overline{u_i' u_j'}}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{5.33}$$

and

$$\overline{u_i' \frac{\partial u_m'}{\partial x_k}} = - \frac{\overline{u_i' u_m'}}{\rho_o} \frac{\partial \rho_o}{\partial x_k} \tag{5.34}$$

Both of these models satisfy the continuity equation (5.4). At the present time, (5.33) is preferred; however, (5.34) was the modeling used in A.R.A.P. Report No. 169 [Ref. 2].

Following the modeling given above, we may give two models for $\overline{T'(\partial u_m'/\partial x_i)}$, namely,

$$\overline{T' \frac{\partial u_m'}{\partial x_i}} = - \delta_{mi} \frac{\overline{T' u_j'}}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{5.35}$$

and

$$\overline{T' \frac{\partial u_m'}{\partial x_i}} = - \frac{\overline{T' u_m'}}{\rho_o} \frac{\partial \rho_o}{\partial x_i} \tag{5.36}$$

and two models for $\overline{C_\alpha'(\partial u_m'/\partial x_i)}$, namely,

$$\overline{C_\alpha' \frac{\partial u_m'}{\partial x_i}} = - \delta_{mi} \frac{\overline{C_\alpha' u_j'}}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{5.37}$$

$$\overline{C_\alpha' \frac{\partial u_m'}{\partial x_i}} = - \frac{\overline{C_\alpha' u_m'}}{\rho_o} \frac{\partial \rho_o}{\partial x_i} \tag{5.38}$$

Here again, the models presently preferred are (5.35) and (5.37) while those used in A.R.A.P. Report No. 169 [Ref. 2] were (5.36) and (5.38).

With the modelings given above, we are in a position to write a closed set of equations for the generation of turbulence in atmospheric shear layers and the dispersal of passive pollutants in these layers.

6. AN INVARIANT MODEL OF THE ATMOSPHERE

If the models given in Section 5 are placed in the set of equations (5.1) through (5.13), a set of closed equations for the motion of the atmosphere and the dispersal of pollutants in this atmosphere is obtained. The equations are not the usual Boussinesq equations, for we have retained the divergence of the mean and turbulent velocity fields, and we have not made the thin layer assumption that is necessary to obtain the Boussinesq approximation. The assumptions that have been made are that the molecular transport coefficients are constant, that the Prandtl and Schmidt numbers are one, and that the departure of the atmosphere from an adiabatic state at rest is small. Since we wish to exhibit the equations used in A.R.A.P. Report No. 169, we will adopt the modelings that were used in that report in the rest of this section. If (5.15) through (5.32), (5.34), (5.36), and (5.38) are substituted into the basic set [(5.1) through (5.13)], one obtains

$$\rho_o \frac{\partial \bar{u}_i}{\partial t} + \rho_o \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = - \frac{\partial \bar{p}}{\partial x_i} + \frac{\rho_o \bar{T}}{T_o} g_i + \mu_o \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\rho_o \overline{u_i' u_j'}) - (\mu_o + \mu_o^*) \frac{\partial}{\partial x_i} \left(\frac{\bar{u}_m}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) \tag{6.1}$$

$$\rho_o \frac{\partial \bar{T}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \bar{T}}{\partial x_j} = \mu_o \frac{\partial^2 \bar{T}}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\rho_o \overline{u_j' T'}) \tag{6.2}$$

$$\frac{\partial \bar{u}_j}{\partial x_j} = - \frac{\bar{u}_j}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{6.3}$$

$$\frac{\partial u_j'}{\partial x_j} = - \frac{u_j'}{\rho_o} \frac{\partial \rho_o}{\partial x_j} \tag{6.4}$$

$$
\begin{aligned}
&\rho_o \frac{\partial \overline{u_i' u_k'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{u_i' u_k'}}{\partial x_j} = \\
&\quad - \rho_o \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j' u_i'} \frac{\partial \bar{u}_k}{\partial x_j} + \frac{\rho_o}{T_o} \overline{T' u_k'}\, g_i + \frac{\rho_o}{T_o} \overline{T' u_i'}\, g_k \\
&\quad + \frac{\partial}{\partial x_j} \left[\rho_o \Lambda_2 q \left(\frac{\partial \overline{u_i' u_j'}}{\partial x_k} + \frac{\partial \overline{u_i' u_k'}}{\partial x_j} + \frac{\partial \overline{u_j' u_k'}}{\partial x_i} \right) \right] \\
&\quad + \frac{\partial}{\partial x_i} \left(\rho_o \Lambda_3 q \frac{\partial \overline{u_k' u_j'}}{\partial x_j} \right) + \frac{\partial}{\partial x_k} \left(\rho_o \Lambda_3 q \frac{\partial \overline{u_i' u_j'}}{\partial x_j} \right) \\
&\quad - \frac{\rho_o q}{\Lambda_1} \left(\overline{u_i' u_k'} - \delta_{ik} \frac{q^2}{3} \right) + \frac{\partial \rho_o}{\partial x_i} \left(\Lambda_3 q \frac{\partial \overline{u_j' u_k'}}{\partial x_j} \right) \\
&\quad + \frac{\partial \rho_o}{\partial x_k} \left(\Lambda_3 q \frac{\partial \overline{u_j' u_i'}}{\partial x_j} \right) + \mu_o \frac{\partial^2 \overline{u_i' u_k'}}{\partial x_j^2} - 2\mu_o \frac{\overline{u_i' u_k'}}{\lambda^2} \\
&\quad - (\mu_o + \mu_o^*) \left\{ \overline{u_i' u_m'} \left[\frac{\partial}{\partial x_k} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) - \frac{1}{\rho_o^2} \frac{\partial \rho_o}{\partial x_m} \frac{\partial \rho_o}{\partial x_k} \right] \right. \\
&\quad \left. + \overline{u_k' u_m'} \left[\frac{\partial}{\partial x_i} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) - \frac{1}{\rho_o^2} \frac{\partial \rho_o}{\partial x_m} \frac{\partial \rho_o}{\partial x_i} \right] \right\} \qquad (6.5)
\end{aligned}
$$

$$\rho_o \frac{\partial \overline{u_i'T'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{u_i'T'}}{\partial x_j} =$$

$$- \rho_o \overline{u_j'T'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j'u_i'} \frac{\partial \bar{T}}{\partial x_j} + \frac{\rho_o}{T_o} \overline{T'^2} g_i$$

$$+ \frac{\partial}{\partial x_j} \left[\rho_o \Lambda_2 q \left(\frac{\partial \overline{u_i'T'}}{\partial x_j} + \frac{\partial \overline{u_j'T'}}{\partial x_i} \right) \right] + \frac{\partial}{\partial x_i} \left[\rho_o \Lambda_3 q \frac{\partial \overline{u_j'T'}}{\partial x_j} \right]$$

$$- \frac{\rho_o q}{\Lambda_1} \overline{u_i'T'} + \mu_o \frac{\partial^2 \overline{u_i'T'}}{\partial x_j^2} - 2\mu_o \frac{\overline{u_i'T'}}{\lambda^2}$$

$$- (\mu_o + \mu_o^*) \left\{ \overline{T'u_m'} \left[\frac{\partial}{\partial x_i} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) - \frac{1}{\rho_o^2} \frac{\partial \rho_o}{\partial x_m} \frac{\partial \rho_o}{\partial x_i} \right] \right\} \tag{6.6}$$

$$\rho_o \frac{\partial \overline{T'^2}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{T'^2}}{\partial x_j} = - 2\rho_o \overline{T'u_j'} \frac{\partial \bar{T}}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\rho_o \Lambda_2 q \frac{\partial \overline{T'^2}}{\partial x_j} \right)$$

$$+ \mu_o \frac{\partial^2 \overline{T'^2}}{\partial x_j^2} - 2\mu_o \frac{\overline{T'^2}}{\lambda^2} \tag{6.7}$$

$$\rho_o \frac{\partial \bar{C}_\alpha}{\partial t} + \rho_o \bar{u}_j \frac{\partial \bar{C}_\alpha}{\partial x_j} = \mu_o \frac{\partial^2 \bar{C}_\alpha}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\rho_o \overline{u_j'C_\alpha'}) \tag{6.8}$$

$$\rho_o \frac{\partial \overline{u_i' C_\alpha'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{u_i' C_\alpha'}}{\partial x_j} =$$

$$- \rho_o \overline{u_j' C_\alpha'} \frac{\partial \bar{u}_i}{\partial x_j} - \rho_o \overline{u_j' u_i'} \frac{\partial \bar{C}_\alpha}{\partial x_j} + \frac{\rho_o}{T_o} \overline{T' C_\alpha'} g_i$$

$$+ \frac{\partial}{\partial x_j} \left[\rho_o \Lambda_2 q \left(\frac{\partial \overline{C_\alpha' u_i'}}{\partial x_j} + \frac{\partial \overline{C_\alpha' u_j'}}{\partial x_i} \right) \right] + \frac{\partial}{\partial x_i} \left(\rho_o \Lambda_3 q \frac{\partial \overline{C_\alpha' u_j'}}{\partial x_j} \right)$$

$$- \frac{\rho_o q}{\Lambda_1} \overline{u_i' C_\alpha'} + \mu_o \frac{\partial^2 \overline{u_i' C_\alpha'}}{\partial x_j^2} - 2\mu_o \frac{\overline{u_i' C_\alpha'}}{\lambda^2}$$

$$- (\mu_o + \mu_o^*) \left\{ \overline{C_\alpha' u_m'} \left[\frac{\partial}{\partial x_i} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_m} \right) - \frac{1}{\rho_o^2} \frac{\partial \rho_o}{\partial x_m} \frac{\partial \rho_o}{\partial x_i} \right] \right\} \quad (6.9)$$

$$\rho_o \frac{\partial \overline{C_\alpha' T'}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{C_\alpha' T'}}{\partial x_j} = - \rho_o \overline{u_j' T'} \frac{\partial \bar{C}_\alpha}{\partial x_j} - \rho_o \overline{u_j' C_\alpha'} \frac{\partial \bar{T}}{\partial x_j}$$

$$+ \frac{\partial}{\partial x_j} \left(\rho_o \Lambda_2 q \frac{\partial \overline{C_\alpha' T'}}{\partial x_j} \right) + \mu_o \frac{\partial^2 \overline{C_\alpha' T'}}{\partial x_j^2} - 2\mu_o \frac{\overline{C_\alpha' T'}}{\lambda^2} \quad (6.10)$$

$$\rho_o \frac{\partial \overline{C_\alpha'^2}}{\partial t} + \rho_o \bar{u}_j \frac{\partial \overline{C_\alpha'^2}}{\partial x_j} = - 2\rho_o \overline{u_j' C_\alpha'} \frac{\partial \bar{C}_\alpha}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\rho_o \Lambda_2 q \frac{\partial \overline{C_\alpha'^2}}{\partial x_j} \right)$$

$$+ \mu_o \frac{\partial^2 \overline{C_\alpha'^2}}{\partial x_j^2} - 2\mu_o \frac{\overline{C_\alpha'^2}}{\lambda^2} \quad (6.11)$$

and, finally,

$$\bar{\rho} = - \frac{\rho_o}{T_o} \bar{T} \quad (6.12)$$

and

$$\rho' = - \frac{\rho_o}{T_o} T' \quad (6.13)$$

Equations (6.1) through (6.13) are the starting point for the model equations given in A.R.A.P. Report No. 169.

7. EQUATIONS USED IN A.R.A.P. REPORT NO. 169

In A.R.A.P. Report No. 169, the equations used in the previous section were written out for a special case of atmospheric motion. If we adopt the convention that (x_1, x_2, x_3) is (x, y, z) and (u_1, u_2, u_3) is (u, v, w) and take the direction z as perpendicular to the earth's surface so that $g_i = \delta_{i3} g$, we may express this special case of atmospheric motion as

$$\begin{aligned}
\bar{u} &= \bar{u}(z,t)\ , \quad \bar{v} = \bar{w} = 0 \\
\bar{T} &= \bar{T}(z,t) \\
\overline{u'u'} &= \overline{u'u'}(z,t) \\
\overline{v'v'} &= \overline{v'v'}(z,t) \\
\overline{w'w'} &= \overline{w'w'}(z,t) \\
\overline{u'w'} &= \overline{u'w'}(z,t) \\
\overline{v'w'} &= 0 \\
\overline{u'v'} &= 0 \\
\overline{u'T'} &= \overline{u'T'}(z,t) \\
\overline{v'T'} &= 0 \\
\overline{w'T'} &= \overline{w'T'}(z,t) \\
\overline{T'^2} &= \overline{T'^2}(z,t)
\end{aligned} \tag{7.1}$$

Equations (6.1) through (6.7) of the previous section may then be written

$$\rho_o \frac{\partial \bar{u}}{\partial t} = - \frac{\partial \bar{p}}{\partial x} + \mu_o \frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial}{\partial z} (\rho_o \overline{u'w'}) \tag{7.2}$$

$$\frac{\partial \bar{p}}{\partial y} = 0 \tag{7.3}$$

$$\frac{\partial \bar{p}}{\partial z} = \frac{\rho_o}{T_o} g\bar{T} - \frac{\partial}{\partial z}\left(\rho_o \overline{w'w'}\right) \tag{7.4}$$

$$\rho_o \frac{\partial \bar{T}}{\partial t} = \mu_o \frac{\partial^2 \bar{T}}{\partial z^2} - \frac{\partial}{\partial z}\left(\rho_o \overline{w'T'}\right) \tag{7.5}$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{7.6}$$

$$\frac{\partial u_i'}{\partial x_i} = - \frac{w'}{\rho_o} \frac{\partial \rho_o}{\partial z} \tag{7.7}$$

$$\begin{aligned} \rho_o \frac{\partial \overline{u'u'}}{\partial t} = &- 2\rho_o \overline{u'w'} \frac{\partial \bar{u}}{\partial z} + \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{u'u'}}{\partial z}\right) \\ &- \frac{\rho_o q}{\Lambda_1}\left(\overline{u'u'} - \frac{q^2}{3}\right) \\ &+ \mu_o \frac{\partial^2 \overline{u'u'}}{\partial z^2} - 2\mu_o \frac{\overline{u'u'}}{\lambda^2} \end{aligned} \tag{7.8}$$

$$\begin{aligned} \rho_o \frac{\partial \overline{v'v'}}{\partial t} = &\frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{v'v'}}{\partial z}\right) - \frac{\rho_o q}{\Lambda_1}\left(\overline{v'v'} - \frac{q^2}{3}\right) \\ &+ \mu_o \frac{\partial^2 \overline{v'v'}}{\partial z^2} - 2\mu_o \frac{\overline{v'v'}}{\lambda^2} \end{aligned} \tag{7.9}$$

$$\begin{aligned} \rho_o \frac{\partial \overline{w'w'}}{\partial t} = &\frac{2\rho_o}{T_o} g\overline{T'w'} + 3 \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{w'w'}}{\partial z}\right) + 2 \frac{\partial}{\partial z}\left(\rho_o \Lambda_3 q \frac{\partial \overline{w'w'}}{\partial z}\right) \\ &- \frac{\rho_o q}{\Lambda_1}\left(\overline{w'w'} - \frac{q^2}{3}\right) + 2\Lambda_3 q \frac{\partial \overline{w'w'}}{\partial z} \frac{\partial \rho_o}{\partial z} \\ &+ \mu_o \frac{\partial^2 \overline{w'w'}}{\partial z^2} - 2\mu_o \frac{\overline{w'w'}}{\lambda^2} \\ &- 2(\mu_o + \mu_o^*)\left\{\overline{w'w'}\left[\frac{\partial}{\partial z}\left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z}\right) - \frac{1}{\rho_o^2}\left(\frac{\partial \rho_o}{\partial z}\right)^2\right]\right\} \end{aligned} \tag{7.10}$$

$$\rho_o \frac{\partial \overline{u'w'}}{\partial t} = - \rho_o \overline{w'w'} \frac{\partial \bar{u}}{\partial z} + \frac{\rho_o}{T_o} g\overline{u'T'}$$

$$+ 2 \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{u'w'}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\rho_o \Lambda_3 q \frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$- \frac{\rho_o q}{\Lambda_1} \overline{u'w'} + \Lambda_3 q \frac{\partial \overline{u'w'}}{\partial z} \frac{\partial \rho_o}{\partial z}$$

$$+ \mu_o \frac{\partial^2 \overline{u'w'}}{\partial z^2} - 2\mu_o \frac{\overline{u'w'}}{\lambda^2}$$

$$- (\mu_o + \mu_o^*)\left\{\overline{u'w'}\left[\frac{\partial}{\partial z}\left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z}\right) - \frac{1}{\rho_o^2}\left(\frac{\partial \rho_o}{\partial z}\right)^2\right]\right\} \tag{7.11}$$

$$\rho_o \frac{\partial \overline{u'T'}}{\partial t} = - \rho_o \overline{w'T'} \frac{\partial \bar{u}}{\partial z} - \rho_o \overline{u'w'} \frac{\partial \bar{T}}{\partial z}$$

$$+ \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{u'T'}}{\partial z}\right) - \frac{\rho_o q}{\Lambda_1} \overline{u'T'}$$

$$+ \mu_o \frac{\partial^2 \overline{u'T'}}{\partial z^2} - 2\mu_o \frac{\overline{u'T'}}{\lambda^2} \tag{7.12}$$

$$\rho_o \frac{\partial \overline{w'T'}}{\partial t} = - \rho_o \overline{w'w'} \frac{\partial \bar{T}}{\partial z} + \frac{\rho_o}{T_o} g\overline{T'^2}$$

$$+ 2 \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{w'T'}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\rho_o \Lambda_3 q \frac{\partial \overline{w'T'}}{\partial z}\right)$$

$$- \frac{\rho_o q}{\Lambda_1} \overline{w'T'} + \mu_o \frac{\partial^2 \overline{w'T'}}{\partial z^2} - 2\mu_o \frac{\overline{w'T'}}{\lambda^2}$$

$$- (\mu_o + \mu_o^*)\left\{\overline{T'w'}\left[\frac{\partial}{\partial z}\left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z}\right) - \frac{1}{\rho_o^2}\left(\frac{\partial \rho_o}{\partial z}\right)^2\right]\right\} \tag{7.13}$$

$$\rho_o \frac{\partial \overline{T'^2}}{\partial t} = -2\rho_o \overline{w'T'} \frac{\partial \overline{T}}{\partial z} + \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{T'^2}}{\partial z}\right) + \mu_o \frac{\partial^2 \overline{T'^2}}{\partial z^2} - 2\mu_o \frac{\overline{T'^2}}{\lambda^2} \qquad (7.14)$$

In addition, we have

$$\tilde{\rho} = -\frac{\rho_o}{T_o}\overline{T} \qquad (7.15)$$

and

$$\rho' = -\frac{\rho_o}{T_o}T' \qquad (7.16)$$

so that $\bar{\rho}$, $\overline{\rho' u_i'}$, and $\overline{\rho'^2}$ may be found from $\overline{T}$, $\overline{T'u_i'}$, and $\overline{T'^2}$.

The equations given above are the same as those used in A.R.A.P. Report No. 169 if one places $\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda$. In Table 7.1 below, the equation numbers of the two reports are compared.

Table 7.1

Equation Number A.R.A.P. 169	Equation Number This Report
16	7.2
17	7.5
18	7.8
19	7.9
20	7.10
21	7.11
22	7.12
23	7.13
24	7.14

In A.R.A.P. Report No. 169, the turbulent structure of an atmospheric shear layer having given time-independent distributions $\bar{u}(z)$ and $\overline{T}(z)$ was found by solving (7.8) through (7.14) simultaneously for the distributions of $\overline{u_i'u_k'}$, $\overline{u_i'T'}$, and $\overline{T'^2}$ in z which came to equilibrium with the given profiles $\bar{u}(z)$ and $\bar{T}(z)$. Once these distributions were known, the transport of a pollutant species C_α in such a layer from a steady line or point source was discussed in terms of Eqs. (6.8) through (6.11).

To get the equations used in A.R.A.P. Report No. 169, the mean fields $\bar{u} = \bar{u}(z)$, $\bar{T} = \bar{T}(z)$, $\overline{u'u'}(z)$, $\overline{v'v'}(z)$, $\overline{w'w'}(z)$, $\overline{u'w'}(z)$, $\overline{u'T'}(z)$, $\overline{w'T'}(z)$, and $\overline{T'^2}(z)$ were assumed given. In addition, $\overline{u'v'}$, $\overline{v'w'}$, and $\overline{v'T'}$ were equal to zero. The source was assumed to be steady; therefore, one sought $\bar{C}_\alpha(x,y,z)$. To simplify the problem, the thin layer approximation was used so that the derivative of a quantity in the streamwise direction x could be neglected compared to the derivative of that same quantity with respect to y and z. Under this assumption, (6.8) through (6.11) can be written

$$\rho_o \bar{u} \frac{\partial \bar{C}_\alpha}{\partial x} = \mu_o \frac{\partial}{\partial y}\left(\frac{\partial \bar{C}_\alpha}{\partial y} - \rho_o \overline{v'C'_\alpha}\right) + \mu_o \frac{\partial}{\partial z}\left(\frac{\partial \bar{C}_\alpha}{\partial z} - \rho_o \overline{w'C'_\alpha}\right) \quad (7.17)$$

$$\begin{aligned}\rho_o \bar{u} \frac{\partial \overline{C'_\alpha v'}}{\partial x} = &- \rho_o \overline{v'v'} \frac{\partial \bar{C}_\alpha}{\partial y} + \frac{\partial}{\partial y}\left[\rho_o (2\Lambda_2 + \Lambda_3) q \frac{\partial \overline{C'_\alpha v'}}{\partial y}\right] \\ &+ \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{C'_\alpha v'}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{C'_\alpha w'}}{\partial y}\right) \\ &+ \frac{\partial}{\partial y}\left(\rho_o \Lambda_3 q \frac{\partial \overline{C'_\alpha w'}}{\partial z}\right) - \frac{\rho_o q}{\Lambda_1} \overline{C'_\alpha v'} \\ &+ \mu_o \left(\frac{\partial^2 \overline{C'_\alpha v'}}{\partial y^2} + \frac{\partial^2 \overline{C'_\alpha v'}}{\partial z^2} - 2 \frac{\overline{C'_\alpha v'}}{\lambda^2}\right) \quad (7.18)\end{aligned}$$

$$
\begin{aligned}
\rho_o \bar{u} \frac{\partial \overline{C'_\alpha w'}}{\partial x} = &- \rho_o \overline{w'w'} \frac{\partial \bar{C}_\alpha}{\partial z} + \frac{\rho_o g}{T_o} \overline{C'_\alpha T'} \\
&+ \frac{\partial}{\partial y} \left(\rho_o \Lambda_2 q \frac{\partial \overline{C'_\alpha w'}}{\partial y} \right) \\
&+ \frac{\partial}{\partial z} \left[\rho_o (2\Lambda_2 + \Lambda_3) q \frac{\partial \overline{C'_\alpha w'}}{\partial z} \right] \\
&+ \frac{\partial}{\partial z} \left(\rho_o \Lambda_3 q \frac{\partial \overline{C'_\alpha v'}}{\partial y} \right) + \frac{\partial}{\partial y} \left(\rho_o \Lambda_2 q \frac{\partial \overline{C'_\alpha v'}}{\partial z} \right) \\
&- \frac{\rho_o q}{\Lambda_1} \overline{C'_\alpha w'} \\
&+ \mu_o \left(\frac{\partial^2 \overline{C'_\alpha w'}}{\partial y^2} + \frac{\partial^2 \overline{C'_\alpha w'}}{\partial z^2} - 2 \frac{\overline{C'_\alpha w'}}{\lambda^2} \right) \\
&- (\mu_o + \mu_o^*) \left\{ \overline{C'_\alpha w'} \left[\frac{\partial}{\partial z} \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z} \right) - \frac{1}{\rho_o^2} \left(\frac{\partial \rho_o}{\partial z} \right)^2 \right] \right\}
\end{aligned}
\tag{7.19}
$$

$$
\begin{aligned}
\rho_o \bar{u} \frac{\partial \overline{C'_\alpha T'}}{\partial x} = &- \rho_o \overline{w'T'} \frac{\partial \bar{C}_\alpha}{\partial z} - \rho_o \overline{w'C'_\alpha} \frac{\partial \bar{T}}{\partial z} \\
&+ \frac{\partial}{\partial y} \left(\rho_o \Lambda_2 q \frac{\partial \overline{C'_\alpha T'}}{\partial y} \right) + \frac{\partial}{\partial z} \left(\rho_o \Lambda_2 q \frac{\partial \overline{C'_\alpha T'}}{\partial z} \right) \\
&+ \mu_o \left(\frac{\partial^2 \overline{C'_\alpha T'}}{\partial y^2} + \frac{\partial^2 \overline{C'_\alpha T'}}{\partial z^2} - 2 \frac{\overline{C'_\alpha T'}}{\lambda^2} \right)
\end{aligned}
\tag{7.20}
$$

$$\rho_o \bar{u} \frac{\partial \overline{C_\alpha'^2}}{\partial x} = -2\rho_o \overline{w'C_\alpha'} \frac{\partial \bar{C}_\alpha}{\partial z} + \frac{\partial}{\partial y}\left(\rho_o \Lambda_2 q \frac{\partial \overline{C_\alpha'^2}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho_o \Lambda_2 q \frac{\partial \overline{C_\alpha'^2}}{\partial z}\right) + \mu_o \left(\frac{\partial^2 \overline{C_\alpha'^2}}{\partial y^2} + \frac{\partial^2 \overline{C_\alpha'^2}}{\partial z^2} - 2\frac{\overline{C_\alpha'^2}}{\lambda^2}\right) \quad (7.21)$$

We note here that (7.17) through (7.20) are exactly the same equations as given in A.R.A.P. Report No. 169 (Eqs. (12) through (15)), if one writes $\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda$.

8. DISCUSSION

The equations given in the previous section, and which were used in A.R.A.P. Report No. 169, may be simplified by considering high Reynolds number arguments. Normally, all terms containing the viscosities μ_o or μ_o^* can be dropped from the equations, except for the dissipation term which is of the form

$$\overline{\frac{\partial a'}{\partial x_j} \frac{\partial b'}{\partial x_j}} = -2\mu_o \frac{\overline{a'b'}}{\lambda^2} \tag{8.1}$$

This term may not be dropped because the equation for the dissipative scale λ is viscosity- or Reynolds number-dependent and is of the form [see Ref. 5]

$$\frac{1}{\lambda^2} = \frac{a + b(\rho_o q \Lambda_1/\mu_o)}{\Lambda_1^2} \tag{8.2}$$

Thus, at high Reynolds numbers, the dissipation (8.1) is of the form

$$-\frac{2b\rho_o q}{\Lambda_1} \overline{a'b'} \tag{8.3}$$

and may not be dropped from the equations.

In our work we have chosen to retain the viscous terms to enable a shear layer solution to be extended all the way to a smooth solid boundary. It is necessary to retain most of the viscous terms if such a solution is to be obtained, because one must extend the solution through the viscous sublayer near the surface where the viscous terms are the dominant ones in the equations.

A simplification is possible, however. Since the terms which appear in the equations that are multiplied by the sum $(\mu_o + \mu_o^*)$ are caused by the inclusion of the divergence of the velocity field (which may certainly be neglected in the sublayer), these terms may, for atmospheric flows, be dropped from the equations.

If the terms containing $(\mu_o + \mu_o^*)$ are dropped, it is evident (since these terms were the ones that contained the divergence/strain effects) that the problem of modeling such terms is eliminated. Thus, the only new term that one is required to model in considering these non-Boussinesq atmospheric equations compared with the Boussinesq model is the divergence that shows up in the tendency-towards-isotropy term in the equation for $\overline{u_i'u_k'}$. As mentioned in Section 5 where modeling was discussed in some detail, we are considering the replacement of the model used in Ref. 1 and in A.R.A.P. Report No. 169, i.e., (5.15)

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} = -\frac{\rho_o q}{\Lambda_1}\left(\overline{u_i'u_k'} - \delta_{ik}\frac{q^2}{3}\right) - \frac{\overline{p'u_k'}}{\rho_o}\frac{\partial \rho_o}{\partial x_i} - \frac{\overline{p'u_i'}}{\rho_o}\frac{\partial \rho_o}{\partial x_k}$$

with the simpler model (5.14)

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} = -\frac{\rho_o q}{\Lambda_1}\left(\overline{u_i'u_k'} - \delta_{ik}\frac{q^2}{3}\right) - 2\delta_{ik}\frac{\overline{p'u_m'}}{\rho_o}\frac{\partial \rho_o}{\partial x_m}$$

Since the effect of including the extra term

$$-\left(\frac{\overline{p'u_k'}}{\rho_o}\frac{\partial \rho_o}{\partial x_i} + \frac{\overline{p'u_i'}}{\rho_o}\frac{\partial \rho_o}{\partial x_k}\right)$$

or

$$-2\delta_{ik}\frac{\overline{p'u_m'}}{\rho_o}\frac{\partial \rho_o}{\partial x_m}$$

in the model for the tendency towards isotropy does not, in most cases, have a significant effect upon the turbulent velocity field, we do not intend to modify the equations in Section 7 (which are currently programmed on the computer at A.R.A.P.) until such time as a detailed reëxamination of the entire invariant model that is in use at the present time is made.

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15. Supplementary Notes

16. Abstracts

A detailed derivation of the equations which describe the generation of turbulence in the atmosphere and the equations which govern the transport of pollutants in the atmosphere is given. The equations are valid for motions which are small departures from an atmosphere at rest and in adiabatic equilibrium, but the usual assumption that the motion is restricted to thin layers, which leads to the Boussinesq approximation, is relaxed.

17. Key Words and Document Analysis. 17a. Descriptors

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Turbulence models
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Invariant modeling

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