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# Mathematical Problem Solving

Sponsored by the Georgia Center  
for the Study of Learning and  
Teaching Mathematics  
and the  
Department of Mathematics Education  
University of Georgia  
Athens, Georgia

Larry L. Hatfield, Editor  
David A. Bradbard, Technical Editor

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MATHEMATICAL

PROBLEM

SOLVING

Papers from a Research Workshop

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ERIC Clearinghouse for Science, Mathematics,  
and Environmental Education  
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## MATHEMATICS EDUCATION REPORTS

The Mathematics Education Reports series makes available recent analyses and syntheses of research and development efforts in mathematics education. We are pleased to make available as part of this series the papers from the Workshop on Mathematical Problem Solving sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics.

Other Mathematics Education Reports make available information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education.

Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by our Clearinghouse.

Jon L. Higgins  
Associate Director for  
Mathematics Education

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## Acknowledgements and Overview

The Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM) was started July 1, 1975, through a founding grant from the National Science Foundation. Various activities preceded the founding of the GCSLTM. The most significant was a conference held at Columbia University in October of 1970 on Piagetian Cognitive-Development and Mathematical Education. This conference was directed by the late Myron F. Roszkopf and jointly sponsored by the National Council of Teachers of Mathematics and the Department of Mathematical Education, Teachers College, Columbia University with a grant from the National Science Foundation. Following the October 1970 Conference, Professor Roszkopf spent the winter and spring quarters of 1971 as a visiting professor of Mathematics Education at the University of Georgia. During these two quarters, the editorial work was accomplished on the proceedings of the October conference and a Letter of Intent was filed in February of 1971 with the National Science Foundation to create a Center for Mathematical Education Research and Innovation. Professor Roszkopf's illness and untimely death made it impossible for him to develop the ideas contained in that Letter.

After much discussion among faculty in the Department of Mathematics Education at the University of Georgia, it was clear that a center devoted to the study of mathematics education ought to attack a broader range of problems than was stated in the Letter of Intent. As a result of these discussions, three areas of study were identified as being of primary interest in the initial year of the Georgia Center for the Study of Learning and Teaching Mathematics--Teaching Strategies, Concept Development, and Problem Solving. Thomas J. Cooney assumed directorship of the Teaching Strategies Project, Leslie P. Steffe the Concept Development Project, and Larry L. Hatfield the Problem Solving Project.

The GCSLTM is intended to be a long-term operation with the broad goal of improving mathematics education in elementary and secondary schools. To be effective, it was felt that the Center would have to include mathematics educators with interests commensurate with those of the project areas. Alternative organizational patterns were available--resident scholars, institutional consortia, or individual consortia. The latter organizational pattern was chosen because it was felt maximum participation would be then possible. In order to operationalize a concept of a consortia of individuals, five research workshops were held during the spring of 1975 at the University of Georgia. These workshops were (ordered by dates held) Teaching Strategies, Number and Measurement Concepts, Space and Geometry Concepts, Models for Learning Mathematics,

and Problem Solving. Papers were commissioned for each workshop. It was necessary to commission papers for two reasons. First, current analyses and syntheses of the knowledge in the particular areas chosen for investigation were needed. Second, catalysts for further research and development activities were needed--major problems had to be identified in the project areas on which work was needed.

Twelve working groups have emerged from these workshops, three in Teaching Strategies, five in Concept Development, and four in Problem Solving. The three working groups in Teaching Strategies are: Differential Effects of Varying Teaching Strategies, John Dossey, Coordinator; Development of Protocol Materials to Depict Moves and Strategies, Kenneth Retzer, Coordinator; and Investigation of Certain Teacher Behavior That May Be Associated with Effective Teaching, Thomas J. Cooney, Coordinator. The five working groups in Concept Development are: Measurement Concepts, Thomas Romberg, Coordinator; Rational Number Concepts, Thomas Kieren, Coordinator; Cardinal and Ordinal Number Concepts, Leslie P. Steffe, Coordinator; Space and Geometry Concepts, Richard Lesh, Coordinator; and Models for Learning Mathematics, William Geeslin, Coordinator. The four working groups in Problem Solving are: Instruction in the Use of Key Organizers (Single Heuristics), Frank Lester, Coordinator; Instruction Organized to use Heuristics in Combinations, Phillip Smith, Coordinator; Instruction in Problem Solving Strategies, Douglas Grouws, Coordinator; and Task Variables for Problem Solving Research, Gerald Kulm, Coordinator. The twelve working groups are working as units somewhat independently of one another. As research and development emerges from working groups, it is envisioned that some working groups will merge naturally.

The publication program of the Center is of central importance to Center activities. Research and development monographs and school monographs will be issued, when appropriate, by each working group. The school monographs will be written in nontechnical language and are to be aimed at teacher educators and school personnel. Reports of single studies may be also published as technical reports.

All of the above plans and aspirations would not be possible if it were not for the existence of professional mathematics educators with the expertise in and commitment to research and development in mathematics education. The professional commitment of mathematics educators to the betterment of mathematics education in the schools has been vastly underestimated. In fact, the basic premise on which the GCSLTM is predicated is that there are a significant number of professional mathematics educators with a great deal of individual commitment to creative scholarship. There is no attempt on the part of the Center to buy this scholarship--only to stimulate it and provide a setting in which it can flourish.

The Center administration wishes to thank the individuals who wrote the excellent papers for the workshops, the participants who made the workshops possible, and the National Science Foundation for supporting financially the first year of Center operation. Various individuals have provided valuable assistance in preparing the papers given at the workshops for publication. Mr. David Bradbard provided technical editorship; Mrs. Julie Wetherbee, Mrs. Elizabeth Platt, Mrs. Kay Abney, and Mrs. Cheryl Hirstein, proved to be able typists; and Mr. Robert Petty drafted the figures. Mrs. Julie Wetherbee also provided expertise in the daily operation of the Center during its first year. One can only feel grateful for the existence of such capable and hardworking people.

Thomas J. Cooney  
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## Research on Mathematical Problem Solving:

### An Overview

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As human endeavors go, it is a complex task to help someone else to become a better problem solver. Yet teachers, parents, and even children routinely engage in this task. And their efforts yield some degree of success with most learners--witness the increasing complexity of society and the solving skills necessary to cope with and advance civilization. But it is still largely a mystery why certain efforts with certain learners seem to produce either lesser or greater results. Attempts to describe why or how a person solves a mathematical problem have resulted in rather shallow, primitive and incomplete pictures. Explanations of how mathematical problem-solving competence builds across a person's experiences are similarly thin. Predictive theories of human problem solving are non-existent.

Most mathematics educators consider any learning goals relative to problem solving to be of major importance. The contributions which the researcher might make to these goals require careful deliberation and planning. While this may have been also necessary in the past, the research enterprise in mathematical education has not always built upon a thoughtful analysis of the researchable problems and scholarly methods of solution for studying the daily functioning of classroom instruction and learning of mathematics.

It appears that tremors of change may be assuming quake proportions within certain sectors of educational research. Reconstructionists, such as Cronbach (1966,1975), Shulman (1970), Snow (1974), and Magoon (1977), are calling for educational researchers to adopt philosophies and methodologies that require a break from contrived, laboratory-oriented settings. They are advocating that the dominant context of formal education--the classroom with groups of students studying standard schooling subjects--must again become the experimental environment. Current psychological theories of learning are incapable of explaining or directing activities in such classrooms. Yet to the leading American psychologists at the turn of this century the study of the educational process in classrooms was the vital focus of their discipline. The subsequent rejection of all unobservable mental processes, which characterized the transformation of psychology during the antimentalistic revolution, led to the psychologists' retreat to ivory-tower laboratories and to non-human subjects.

To a large extent the infant field of mathematics education research continues to emulate the tactics and academic standards imposed by this Fisherian tradition. Prototypic studies are usually short, narrowly circumscribed, focussed on behavioral outcomes, and quickly analyzed. Though perhaps conducted in a classroom, the research methodology usually ignores most of the complexity of that environment by intentionally ignoring the constructive processes or the situational (environmental) variables. Shulman (1970) discussed several aspects in the study of educational environments. He observed that if research is conducted in a setting with characteristics similar to school situations, then one may be able to make reasonable extrapolations to the classroom milieu. He urged a renewed concern for external validity so that "the experimental conditions can serve as a sample from which to make inferences to a population of external conditions of interest" (1970, p.377). In order to study the characteristics and effects of educational environments, he recommended a "distinctive features" analysis:

To deal with the discontinuity between the settings of research and of educational application, a common language or set of terms for characterizing both experimental educational settings and curricula is needed. Researchers must seriously strive to develop a means of analyzing the characteristics of both experimental and school settings into a complex of distinctive features so the task validity of any particular experiment can be estimated in terms of the particular criterion setting to which inferences are being made...

I envisage ultimately a situation in which use of such a distinctive features approach would allow one to characterize the instructional settings to which a particular body of experimental research would most effectively be applicable. Conversely, one could begin with a curriculum of interest and use such an approach to identify critical experiments that might be conducted to examine particular features of the complex curricular Gestalt. (Shulman, 1970, p. 379)

Cronbach (1966), discussing the logic of experiments on "discovery learning," observed that a particular educational tactic is part of a total instructional system. Placement in a context always in combination with other tactics prohibits conclusions to be drawn about the tactic considered simply by itself. Educational researchers are called upon to study an educational tactic in its proper context. The approaches used by Soviet psychology through the conduct of "teaching experiments" offers an important framework for such studies.

In addition to returning the educational research spotlight onto the classroom, Magoon (1977) advocated refocusing our research on another overlooked aspect. Based upon acceptance of cognitive views in psychology

and sociology, he speculated that a constructivist perspective will likely gain credibility among educational researchers. Constructivism assumes that knowledge is a phenomenon built-up within and by the human subject. The mechanisms and processes for such constructions are crucial to an understanding of their knowledge and consequently for interpreting the behaviors and actions of a subject. He proposed that approaches to the study of such phenomena must be primarily ethnographic, involving extensive descriptive and interpretive efforts at explaining the complexity. Cronbach (1975), offering explanations for the absence of strong experimental evidence of aptitude-treatment interactions in school settings, recommended a similar methodological shift: researchers reverse the priority from building generalizations about variables to attending to each particular situation and the localized effects along with any factors unique to that locale. Scriven (1972) also suggested a general relaxation of the constrained traditional experimentalism. In examining the traditional concept of "reliability" he noted that if the usual objections to self-reports were overcome, educational researchers would naturally pay more attention to people's reasons for action in contrast to their present attempts to determine causal accounts of it.

What this suggests for research on mathematical problem solving must be decided by the interpretations of scholars in our field. We do not lack for ideas and numerous successes in the teaching of a variety of problematic emphases to differing children. Perhaps we do need to open up to new perspectives in the study of problem-solving instruction and learning with alternative methodologies, such as ethnographic, anthropologic, and even artistic. We need to shift attention to longitudinal case studies. New emphases on situational variables and analysis of environments should be assumed. Many scholarly, artistic mathematics educators, essentially "turned-off" by the Fisherian tradition of experimentalism, would find new acceptance as researchers.

The papers of this monograph present differing but compatible perspectives for investigating mathematical problem solving. Kilpatrick offers a careful analysis of variables and methodologies for research on problem solving. Adopting the traditional separation of dependent and independent variables, he presents a thorough typology for studies of learning and for studies of teaching problem solving. In his suggestions for methodologies he also promotes "intensive study of the same classrooms over an extended period of time."

Following the theme for the Research Workshop of "instruction in heuristical methods," Hatfield reviews pedagogical rationales and recent studies in this area. Several general qualities of needed research are offered toward the planning of future investigations.

Kantowski's paper describes the Soviet "teaching experiment" and its origin in the U.S.S.R. Specific suggestions for its potential use in investigations on mathematical problem solving are presented. The paper by Lester offers an extensive review of research efforts directed particularly at the elementary school levels. The activities of the tri-site

Mathematical Problem Solving Project, based at Indiana University, are described as a context for several interesting research results and questions. Since the writing of Lester's paper the project has been discontinued, but several of the project's participants have continued with their research efforts in this area.

The Problem Solving Research Workshop served to provide an initial impetus to the formation of an intellectual consortium of researchers of mathematical problem solving. The productive efforts of the participants during the intervening years has led to solidifying the first stages of collaborative research. Optimistically, new conceptions and results will be forthcoming.

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# Variables and Methodologies in Research on Problem Solving\*

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The primary consideration a researcher ought to keep in mind as he plans and conducts a study is: What am I trying to find out by doing this study? Anyone conducting research on problem solving in mathematics needs to be especially clear about the purpose of a study since there is so much unexplored territory in which to get lost.

Only when one has the purpose clearly formulated is it appropriate to ask what variables are involved in the study, whether additional variables should be considered, and what methodology or methodologies should be used to gather data on these variables. Although it makes little sense to choose a variable or methodology before one has settled on the research question to be investigated, there may be some value in discussing the kinds of variables and methodologies that are available—as an aid to the researcher who has framed a question but who has not yet decided how to investigate it. This paper is an attempt to survey some variables and methodologies that one might use in research on problem solving in mathematics, with particular attention to those that appear most promising. Research on problem solving per se is considered separately from research on the teaching of heuristics.

## Variables in Research on Problem Solving in Mathematics

Variables can be classified in a variety of ways, depending on one's purpose. Classifications include stimulus variables, response variables, and intervening variables (Travers, 1964), and active variables versus assigned variables (Ary, Jacobs & Razavieh, 1972). The most common scheme, borrowed from mathematics and science, is to classify a variable in a research study as independent or dependent. In the narrow sense, "independent variable" refers to the condition manipulated in an experiment. (One should recall the admonition of David Hawkins, 1966, that "to call something an independent variable is not to use a name but to claim an achievement [p. 6].") In the broad sense, however, a variable is classed as independent if it is used in making predictions. Variables referring to the behavior being predicted are called dependent variables. A given variable can be independent or dependent, depending on its role in the study, but most variables tend to be used in only one way.

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### Independent Variables

Any study of problem solving in mathematics involves a person (subject) solving a mathematical problem (task) under some condition (situation). Each of these components can be used to define a class of variables.

### Subject Variables

Subject variables can be categorized according to whether or not they are based on a sample of the subject's behavior. Some variables describe the subject as a person: his sex, height, educational status, etc. Such variables can be measured by direct observation, examination of records, or the subject's own report. Other variables are based on inferences derived from a sample of the subject's behavior, say, in response to a test or questionnaire or as observed by a clinician. Such variables include various kinds of aptitudes, abilities, attitudes, and achievement.

A related, and perhaps more useful, classification of subject variables is based on the extent to which they can be modified experimentally. Variables not open to such modification are sometimes termed "organismic" or "assigned" variables. Variables open to some modification (and requiring a sample of behavior) may be termed "trait" variables. (The last two terms are not standard and perhaps not even satisfactory since such traits as general mental ability are considered all but unmodifiable in most research contexts, and not all modes of modification can be termed "instruction." For the purpose of this paper, however, the terms will suffice.)

Organismic variables. In research on problem solving in mathematics, information on organismic variables such as age, sex, race, and social class may be gathered to assist in describing the sample, but with the exception of age and sex, such variables are seldom used as dependent variables in the design (and then typically in subsidiary hypotheses only). Samples of different ages are sometimes drawn in studies of developmental change, but inferences about the development of problem-solving proficiency must be especially tentative in view of the large role instruction apparently plays. Researchers often must weigh the advantages of a sample of young, inexperienced subjects (so that one can study problem-solving processes in a relatively pristine form, with greater opportunity for detecting "developmental" changes) against the advantages of a sample of older, more experienced subjects (so that one can study a greater variety of more sophisticated processes).

Trait variables. Traits include abilities (such as spatial visualization ability or memory for problems); attitudes, interests, and values (such as attitude toward mathematics or interest in proving theorems); and other personality variables relating to perceptual style,

cognitive style, self-concept, persistence, anxiety, need for achievement, sociability, etc. Except for the variables dealing with style, which by definition refers to a consistency in behavior across a wide class of situations, each kind of trait variable ranges from the general to the specific (as the examples in parentheses above are meant to suggest). A general trait such as persistence, for example, not only is likely to be less manipulatable than its more specific cousin persistence in solving problems of type X, but is also likely to be less strongly related to ability in the processes used in solving problems of type X. Although ultimately one wants—for purposes of a more powerful theory—to link problem-solving process abilities to variables that are as general as possible, one is probably best advised to begin with variables of some specificity.

As an example, consider the ability to estimate. Psychologists tend to think of this ability—when they think of it at all—as general and presumably unitary. A study by Paull (1971), however, suggested not only that there are distinct estimation abilities in mathematics but also that they may relate in different ways to problem-solving performance.

Abilities relating to memory, classification, generalization, estimation, judgment, verification, and the like are required in the solution of a complex mathematical problem (see Krutetskii, 1976, for a delineation of these and other abilities). Researchers should consider including measures of such abilities in their studies of problem solving, but again the measures should be specific to the phenomena being studied (memory for problems, say, as opposed to associative memory) if a choice between general and specific measures must be made.

Trait variables that seem to have particular promise of being associated with problem-solving performance in mathematics include the ability to generalize a relationship from a small number of instances, the ability to classify problems according to their mathematical structure, the ability to recall structural features of a problem, the ability to estimate the magnitude of a numerical solution, the ability to detect extraneous and insufficient data, a resistance to fatigue in performing mathematical tasks, a sensitivity to problem situations, a preference for elegance in problem solutions, a reflective cognitive style, and a field independent cognitive style. For most of these variables, measuring instruments need to be developed and refined much further. Two categories of trait variables that might be explored in relation to mathematical problem solving are individual differences in brain hemisphere functions (Wittrock, 1974) and in ability to handle semantic versus syntactic processes (Simon, 1975).

Instructional history variables. The instructional history of the subjects—the topics they have studied, the problems they have attempted previously, the techniques of problem solving they have been taught, the types of instruction they have received—generates a set of variables that can be used in describing the sample of subjects, used in selecting the sample, or (as treatment variables—see below) manipulated



as part of the study. Such variables are clearly relevant to the problem-solving process, but they have seldom been considered explicitly except in studies comparing the relative effectiveness of two or more instructional methods. Failure to take account of variation in prior instruction may account for some of the failure to find differences between methods. Even in studies that do not involve treatment comparisons, specification of instructional history variables is likely to assist in the interpretation of results.

### Task Variables

A simple-minded classification of problem tasks (Kilpatrick, 1969) is according to content and structure. Both of these categories bear further examination and elaboration.

Context variables. Suppose two mathematical problems involve the same numbers in the same relation but one deals with rabbits and chickens in a barnyard, the other with two boats on a river. Most people would agree that the two problems are the same in (mathematical) structure, but what word expresses their difference? "Content" might seem suitable at first, but on reflection, "context" appears marginally better since it avoids the connotation that the mathematical content is different.

Whatever the term, the semantic variables characterizing the differences between the physical situation modeled in the problem, as well as the syntactic variables characterizing the language in which the problem is expressed, need to be explored both analytically and empirically.

Structure variables. The issue of problem structure is also more complicated than first thought might suggest. Consider the following problem:

Find the volume  $F$  of the frustum of a right pyramid with square base, given the altitude  $h$  of the frustum, the length  $a$  of a side of its upper base, and the length  $b$  of a side of its lower base.

One way to characterize the structure of this problem is to say that the formula

$$F = \frac{a^2 + ab + b^2}{3}$$

expresses the relation among problem elements and that any other problem in which elements  $a$ ,  $b$ , and  $h$  are in this relation, with  $F$ ,  $a$ , and  $b$  given, has the same structure as this problem. The two problems might be said to have the same syntactic structure.

Another way to characterize the structure of the problem is in terms of the network of all possible steps from one state of the problem to another: the state-space (Goldin & Luger, 1975). For the problem above, a sketch of the state-space is given on the inside front cover of Volume 2 of Mathematical Discovery (Polya, 1965). Another problem having the same state-space might be said to have the same semantic structure. Or perhaps the distinction is better expressed as the structure of the problem (in terms of its mathematical formulation) versus the structure of the problem space (in terms of the set of all possible steps in solving the problem).

Again, regardless of the terminology adopted, the underlying ideas deserve consideration. A host of research problems revolve about the issue of structure: Can subjects classify problems according to structure? Is there an advantage to training subjects to make such classifications? What problem features facilitate transfer across problems differing in context but not structure? Until some dimensions of structure are identified more clearly, the effects of similarities and differences in problem structure cannot be studied systematically.

Format variables. A problem may be presented to a subject orally or in written form. It may or may not involve the manipulation of some apparatus. The instructions may involve the presentation of rules or boundary conditions to be observed in solving the problem, or the subject may be expected to induce these rules or conditions as part of the problem. The problem may be given all at once or one part at a time. The subject may or may not be given hints or encouragement as he solves the problem. He may be asked to think aloud as he works or to retrospect over the course of his solution. He may or may not be permitted or be encouraged to record scratch work. All these variables can be classed as format variables. They are seldom manipulated systematically in a study since they are not ordinarily of interest to the mathematics educator. By ignoring them, the researcher tacitly assumes they do not affect the relationships he observes. Since all generalizations from research on problem solving need to be validated across situations and data gathering methods, however, variation in problem format can be included as part of the validation.

### Situation Variables

The dividing line between format variables and situation variables is not entirely clear. If a subject were given insufficient or misleading information about a problem, the issue would appear to be one of format. If he were told that the experimenter was interested in how fast he solved the problem, when in fact the experimenter was recording how many errors he made, the issue would appear to be one of situation. Both cases involve instructions, but it seems useful to distinguish between variation in the content of the instructions (format) and variation in the subject's perception of the purpose of the task (situation).

A situation variable concerns the conditions, physical and psychological, under which the subject solves, or attempts to solve, the problem. Situation variables include whether or not the subject volunteers for the study, whether or not he is given extrinsic rewards such as money or grades, whether he works alone or in a group, the time of day at which data are gathered, the presence or absence of distractions, the characteristics and behavior of the interviewer, the nature of the problems given previously and the subject's success with them, and whether or not the subject was told how he did on the previous problem. Like format variables, situation variables themselves are of relatively little interest to the mathematics educator, although they may interest the social or educational psychologist. They are nuisance variables—since they will not go away, one can only hope they will not make much difference. Unfortunately, they may.

### Dependent Variables

Some dependent variables are derived from the subject's responses to a problem task; others require additional samples of behavior. Let us consider the latter type first. (Recall that the studies under discussion concern problem solving per se; studies involving instruction in heuristics are yet to be considered.)

### Concomitant Variables

Any of the trait variables mentioned previously may be used as a dependent variable in a problem-solving study. For example, one might investigate how a subject's classification of problems changed after he had solved a set of problem tasks. Or one might ask whether his attitude toward problem solving had changed.

While working on the problem tasks, the subject may have acquired new knowledge of or skill in mathematics beyond simply learning how to solve the problems. Measures of this knowledge or skill could also serve as dependent variables.

As before, one would expect that the more specific the trait, knowledge, or skill, the greater the likelihood it will be influenced by the problem tasks. Experience suggests that one cannot expect much change in a concomitant variable, however specific, if the number of tasks is small.

### Product Variables

Product variables are based on dimensions of the subject's solution to the problem: its correctness, its completeness, its elegance and economy, the speed with which it was attained, and the number and

diversity of the alternative solutions the subject finds. Speed and correctness are the most commonly used product variables, but others ought to be considered if possible.

### Process Variables

Process variables are based on the solution path the subject takes; they are derived from either the subject's verbal report of his thinking or the manipulations he makes with an apparatus. Process variables relate to such things as the subject's strategy (as inferred from the sequence of steps he takes), the heuristics he uses, the algorithms he uses, the efficiency of his solution path, the extent of his perseveration in blind alleys, the nature and number of the errors he makes, and his response to hints.

Any respectable study of problem solving in mathematics should include measures of process variables. Almost nothing is known about the generalizability of these variables across problems and occasions although the words "strategy" and "style" connote such generalizability (Branca & Kilpatrick, 1972). Research on subjects' consistency in their use of process variables would be a valuable contribution.

### Evaluation Variables

It would be nice to have a map of a subject's cognitions after he has solved a problem. How does he view the problem? How does he relate it to other problems he has solved, other information he possesses? Is he aware of the processes he used and the errors he made? How confident is he of his solution? Such questions may be difficult for subjects to answer directly. Considerable ingenuity may be needed in devising instruments to get at the subject's cognitive (and affective) map of the problems he has solved. A subject's report of what he was trying to do as he solved a problem and how he perceives the problem after having worked on it is no less valuable for being subjective. His report provides data that can be obtained in no other way. There is no particular virtue in labeling such data as unscientific and ignoring them.

## Variables in Research on the Teaching of Heuristics in Mathematics

Up to this point the only "treatment" involved in the research studies discussed has been the administration of problems to be solved. Consequently, the distinction between independent and dependent variables has necessarily been made in the broad sense of predictor versus behavior predicted rather than the narrow sense of condition manipulated versus outcome observed. In studies of the

teaching of heuristics, however, one has full-fledged treatment variables; namely, the methods, materials, and other conditions of instruction. Whether or not these treatment variables are manipulated experimentally, they must be considered in designing such studies.

### Independent Variables

The same independent variables considered earlier can appear in research on heuristics. The subject variables remain as before. The task variables can be used to characterize problems used in instruction. The situation variables are essentially the same, but the researcher may need to give more attention to questions of school climate and organization. (The category of "setting variables" used by Richard Turner, 1976, in discussing research on teaching strategies appears to be roughly equivalent to "situation variables.") Categories of variables that need to be added are instructional treatment variables, classroom activity variables, and teacher variables.

### Instructional Treatment Variables

Some variables characterize an instructional treatment in general: The extent to which the treatment is integrated into ongoing school instruction, whether the treatment is the same for all subjects or individualized, whether the treatment is determined in advance or modified according to the subject's response, the extent to which the treatment tasks resemble the outcome tasks, etc. Other variables refer to one facet or another of the treatment.

Method variables. What heuristics are taught as part of the treatment? In what sequence are they taught? Is the instruction itself heuristic? (open-ended? Socratic? inductive?) Does the teacher illustrate how the heuristics are used? Are students given names for the heuristics? What problems are used in instruction? Are problems grouped by type? Are model problems taught for each type? What problems and solutions are discussed with students? What is the nature of this discussion? Are students given a chance to discuss problems and solutions with other students? Each question implies one or more variables that might be used to characterize instructional methods, and the list of such variables is limited only by one's ingenuity in asking such questions.

As noted elsewhere (Kilpatrick, 1973), methods-comparison studies are typically flawed in both conception and execution. The common failure to define methods operationally in such studies appears even more serious when one considers the manifold ways in which methods can vary. Rather than comparing methods, researchers interested in instruction in heuristics should put their energies into devising the best instructional program they can and then demonstrating in detail how the program

functions and how effective it is in the classroom. The creation, tryout, and revision of program components and instruments for measuring effectiveness are research activities of far greater potential than the comparison of methods.

Materials variables. The line between methods and materials is difficult to draw, but it is probably worthwhile to separate the two—at least conceptually. Materials variables in research on heuristics include the nature of the instructional media used, the devices used to represent problem situations, and the nature of the accompanying prose. Materials variables in themselves are likely to be of little interest to mathematics educators.

### Classroom Activity Variables

Instructional treatments ordinarily involve classroom activity. Some of this activity is in accordance with the researcher's plan, and as such is part of the intended treatment. Much of the activity, however, is not under the researcher's control. Although the activity can ultimately be considered part of the instructional treatment package—as it works out in practice—a separate category of classroom activity variables is useful, if only to permit the researcher to check the variation in activity within a treatment group and the congruence between the actual and the intended treatments. (In the latter case, the classroom activity is functioning as a dependent variable.)

The Teaching Strategies Project of the Georgia Center for the Study of Learning and Teaching Mathematics has been examining one part—albeit an important part—of classroom discourse (see Cooney, 1976). Some idea of the broader field of research in classroom activity is given by Dunkin and Biddle (1974). Despite the rapid expansion of the field, apparently only one researcher (Stilwell, 1967) has done a descriptive study of the teaching of problem solving in mathematics. Further work along this line is needed. Although teachers may not pay much attention to heuristics during instruction, someone should be prepared to describe their activity when they do.

### Teacher Variables

Nothing is more distressing than to hear researchers talk of "the teacher variable"—as though there were only one. If only there were. But teachers differ in age, sex, teaching experience, self-confidence, enthusiasm, philosophy of education, attitude toward mathematics, preference for unstructured classroom activity, and love of children. They also differ in problem-solving ability, problem-solving experience, knowledge of heuristics, interest in problem solving, value placed on instruction in heuristics, willingness to delay in supplying

a solution, and ability to accept and to transcend erroneous solutions. One has only to glance at a list of "teacher competencies" to see the variety of ways teachers can differ. Just as no one knows which teacher competencies, if any, are prerequisite to effective teaching, so no one knows which teacher variables, if any, might predict the learning of heuristics.

The analogous issue of teacher competencies is raised here by design. Nothing would be more fruitless than to attempt a catalog of teacher variables on the chance that some might prove to be good predictors of learning. A much better approach would be to identify teachers who seemed to have had some success in teaching heuristics, to see whether this success held up over time and across situations, and then to explore dimensions of similarity among these teachers and dimensions of contrast between these teachers and others deemed less successful. Only then might one be ready to conjecture some relevant teacher variables.

In most research on instruction in heuristics the role of teacher variables will be either to aid in describing the sample of teachers in the study or to suggest plausible reasons for the differences likely to occur in the performance of students taught by different teachers. Since the sample of teachers is likely to be small, the researcher should be able to gather considerable information about each teacher, which would presumably improve either the description or the conjecture.

### Dependent Variables

The dependent variables in research on heuristics are the same dependent variables discussed earlier, plus some new ones. Classroom activity variables can be taken as dependent variables (more precisely, as process variables) if one wishes to learn how the instructional treatment influenced classroom activity. Additional dependent variables that are product variables include all the various measures one could make of what was learned during instruction. One of the models for mathematics achievement (see Wilson, 1971, for examples) might help in organizing these product variables.

### Methodologies in Research on Problem Solving and the Teaching of Heuristics in Mathematics

The preceding discussion was intended to suggest some variables a researcher designing a study ought to consider and then either ignore, eliminate as variables in the design, control through randomization or matching, or build into the design as independent or dependent variables. The effect of the discussion, however, may have been to overwhelm any reader not already paralyzed by the complexity of empirical research on problem solving. Such research is complex, certainly, but

as long as one keeps coming back to one's research question and asking what variables and methodologies bear on it, the complexity ought to be manageable.

In this paper, methodologies relating to historical research, survey research, and reviews of the literature are not considered. Although such studies can be valuable, they demand special methodologies. Most of the numerous books on research methods in education (such as Ary, Jacobs, & Razavieh, 1972; Isaac & Michael, 1971; Travers, 1964) treat these topics.

Methodologies in research on problem solving and the teaching of heuristics in mathematics are so multi-faceted as to defy classification. Consider two of the facets:

1. Type of comparison or contrast. The researcher may be looking for similarities or differences regarding the same or different subjects' responses to the same or different tasks or treatments on the same or different occasions or under the same or different conditions. Each combination of alternatives implies a somewhat different approach.

2. Method of gathering data. The researcher may administer tests or questionnaires; use an apparatus that presents a problem and either record the subject's response himself or have it recorded mechanically; interview subjects given a problem and asked to think aloud or retrospect; use personality inventories, projective tests, or such techniques as word association, the Q-sort, the semantic differential, or the repertory grid; observe subjects solving problems in the classroom or elsewhere; make video- or audio-tape recordings of classroom activity; rely on teachers or students as observers and possibly confederates; act as a participant observer in a group problem-solving situation; act as the teacher in an instructional situation, keeping a log and using recordings to prompt introspection; use the computer to simulate problem-solving processes from protocols gathered by other means; or use instruments to monitor subjects' physiological processes during problem solving or instruction.

The Cartesian product of all possible types of comparison or contrast with all combinations of data-gathering methods only begins to suggest the variety of methodologies one could employ.

The most promising methodologies for research on problem solving in mathematics are those involving intensive study of the same set of subjects over an extended period of time. The subjects must solve a large number of problems of diverse types in order to permit confident generalizations about the processes they use. Numerous measures of trait variables should be obtained, and control should be exercised over instructional history variables. Case studies of subjects selected because of notable giftedness in mathematics or notable difficulty with mathematics may be particularly useful. Cross-age studies of developmental trends in problem solving may help to suggest process variables that should be studied further, but longitudinal studies are obviously to be preferred.



The most promising methodologies for research on the teaching of heuristics in mathematics are those involving intensive study of the same classrooms over an extended period of time. Of special interest are designs in which the experimenter works with the teacher during the course of an academic term or so, observing the effects of various modifications in instruction and using interviews with students to supplement test and observational data. Such studies should lead to the development of materials for instruction in heuristics. The classroom activities of teachers identified as especially effective in teaching problem-solving techniques should be analyzed and contrasted with the classroom activities of teachers having more ordinary attainments. Studies in which the instruction is programmed to control sources of teacher variation may help to suggest which heuristics are most teachable, but studies involving at least some instruction by teachers should predominate.

Experimental studies in which all variables are under tight control are not likely to be of much value in the present state of our ignorance as to how people solve complex mathematical problems and how they might be led to use heuristic methods. Too much developmental work is needed before experimentation could be effective. For example, instruments and techniques must be developed and validated for assessing most of the variables discussed in this paper.

No one is suggesting that researchers abandon the designs and techniques that have served so well in empirical research. But a broader conception of research is needed, and an openness to new techniques, if studies of problem-solving processes and the teaching of heuristics are to have an impact.

Some years ago a group of researchers gave a battery of psychological tests each summer to mathematically talented senior high school students attending special summer institutes at Florida State University. The scores on the tests were intercorrelated, and some correlation coefficients were significant, some not. Several research reports were published (Kennedy, 1962; Kennedy, Cottrell & Smith, 1963, 1964; Kennedy and the Human Development Staff, 1960; Kennedy, Nelson, Lindner, Turner & Moon, 1960). As Krutetskii (1976) notes, the process of solution did not appear to interest the researchers—yet what rich material could have been obtained from these gifted students if one were to study their thinking processes in dealing with mathematical problems. Why were the students simply given a battery of tests to take instead of being asked to solve mathematical problems? It's a good question.

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Heuristical Emphases in the Instruction of Mathematical  
Problem Solving: Rationales and Research

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The significance of the goal of improving the learner's problem-solving competence within school mathematics is well established. Recommendations for emphasizing problem-solving techniques in teaching mathematics can be found throughout the history of mathematics education (Jones, 1970). However, in spite of an implored need that educators must know much more about using problems to stimulate independent and creative thinking, the teaching and learning of problem solving has only occasionally been investigated by mathematics education researchers.

The general goal of the Problem-Solving Project of the Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM) is to organize and conduct coordinated studies of the conditions for and the effects of learning and instruction of mathematics which emphasizes problem solving. Several distinct areas of study encompassing this goal can be identified. These areas include:

- (a) studies devoted to identification of strategies and processes used in solving various mathematical problems, including a search for aptitudes related to these strategies and processes;
- (b) studies devoted to development of clinical procedures for observing and analyzing mathematical problem-solving behaviors;
- (c) studies devoted to development of instructional procedures aimed at improving a student's problem-solving capabilities;
- (d) studies devoted to development of teacher training procedures to result in deliberate employment of instructional methods aimed at enhancing the growth of problem-solving capabilities of students;
- (e) studies of an expository nature, including analytical developments as well as technical reports and interpretative reports of activities and findings of this problem-solving research and development.

The specific project goals during the first year are: (a) to conduct various investigations in mathematical problem solving, (b) to provide an organized synthesis of the literature in mathematical problem solving appropriate to the investigations underway and to the interests of the investigators, (c) to identify detailed specifications for studies to be undertaken in the second round of the GCSLTM's activities, and (d) to establish a working consortia model for conducting coordinated series of investigations. The conduct of the Problem-Solving Research Workshop represents a major step toward obtaining these first-year goals.

The research on problem-solving behavior to be found in the literature of psychology and education (including mathematical education) is considerably varied. As a matter of project strategy it was decided that the theme of the Problem-Solving Workshop would be "instruction in heuristical methods." While intending to provide a focus and a direction for our initial research efforts particularly into area c (development of instructional procedures), such an emphasis should eventually span the five areas for studies noted earlier.

The purpose of this paper is to contribute a common perspective for conceptualizing investigations reflecting the theme of teaching mathematical problem solving by, and for, heuristical methods. The following sections include a discussion of rationales for emphasizing heuristical precepts in teaching and researching mathematical problem solving, a review of selected recent mathematical education research involving heuristical methods, and an identification of possible directions and dimensions of studies to be undertaken in the Problem-Solving Project.

#### Instruction in Heuristical Methods: What, Why and How

The most widely known contemporary and practical treatment of heuristic is due to the eminent mathematician Polya (1957, 1962, 1965). Recently Wickelgren (1974) has sought to provide an extension of such treatments, blending significant developments from the field of artificial intelligence and information processing models with Polya's maxims. In addition, Higgins (1971) offers an interpretation of heuristic as it applies to a methodology for mathematics instruction termed "heuristic teaching." It must be assumed that Workshop participants are reasonably familiar with these elaborate writings. At the same time a brief review of particular aspects will serve to highlight and clarify certain points of view.

What are "heuristical methods" for problem solving? A synopsis from Polya's discussions is presented in search of clarification.

Heuristic, or heuretic, or "ars inveniendi" was the name of a certain branch of study, not very clearly circumscribed, belonging to logic, or to philosophy, or to psychology, often

outlined, seldom presented in detail, and as good as forgotten today. The aim of heuristic is to study the methods and rules of discovery and invention. (Polya, 1957, p. 112)

Heuristic, as an adjective, means "serving to discover," (Polya, 1957, p. 113)

Modern heuristic endeavors to understand the process of solving problems especially the mental operations typically useful in this process. It has various sources of information none of which should be neglected. A serious study of heuristic should take into account both the logical and psychological background, it should not neglect what such older writers as Pappus, Descartes, Leibnitz and Bolzano have to say about the subject, but it should least neglect unbiased experience. Experience in solving problems and experience in watching other people solving problems must be the basis on which heuristic is built. (Polya, 1957, pp. 129-30)

Heuristic reasoning is reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem. We are often obliged to use heuristic reasoning. We shall attain complete certainty when we shall have obtained the complete solution, but before obtaining certainty we must often be satisfied with a more or less plausible guess. (Polya, 1957, p. 113)

Polya approaches heuristic from a practical teacher-oriented aspect: "I am trying, by all the means at my disposal, to entice the reader to do problems and to think about the means and methods he uses in doing them" (Polya, 1962, p. vi). His detailed "case histories" of problem solutions feature questions and suggestions which, organized into four phases of work on the solution (understanding, planning, carrying out, looking back), have come to be known as his "planning heuristic." According to Polya, these questions and suggestions have two common characteristics, generality (in that they indicate a general direction of action and thus may help unobtrusively) and common sense (in order that they can occur naturally or easily to the solver himself). But the significant assumption by Polya is the following:

If the reader is sufficiently acquainted with the list and can see, behind the suggestion, the action suggested, he may realize that the list enumerates, indirectly, mental operations typically useful for the solution of problems. (Polya, 1957, p. 2)

Thus, "heuristic methods" for solving problems include plausible but uncertain actions of a general yet natural character. To know and to use "heuristic methods" at some level of effectiveness suggests that one also knows in some fashion the cognitive operations bearing on one's own problem-solving behavior.

This notion hints at potentially important, yet typically subtle, distinctions: curriculum or instruction aimed at teaching for problem solving versus teaching about problem solving versus teaching via problem solving. Most contemporary school mathematics textbooks claim to teach for problem-solving outcomes; in the sense that organized, usable knowledge in mathematics (concepts, principles, skills) is essential to being an effective problem solver in mathematics, these claims are partially justified. Few texts, however, seek to teach rather explicitly about problem solving in the sense of heuristic. And correspondingly few textbooks approach content as heuristically oriented problem solving as Polya or Higgins recommend.

What should constitute "instruction in heuristic methods," particularly as it relates to mathematical education, is still an ill-defined matter. Again, one can find various characterizations and exemplifications offered by Polya (1962, 1965), Wickelgren (1974), Covington and Crutchfield (1965), Wilson (1967), Butts (1973), and others. An essential feature seems to be the explicit identification and use of heuristic ploys within the act of solving mathematical problems. Sometimes these are modeled by the teacher or instructional materials to be observed and to be imitated by the learner, while at other times they are to be initiated and to be practiced in the learner's own problem-solving actions. Further discussion on this question will be offered in the final section of this paper relating particular strategies for our research.

Why should mathematics educators choose to give "instruction in heuristic methods" of problem solving? Here again, the cogent arguments offered by Polya (see especially 1957, pp. 1-32; 1965, pp. 99-142) encompass learner motivation, educational relevancy, general culture, enhancing common sense reasoning, and active learning. These arguments must certainly be accompanied now by acceptable research findings regarding the efficacy of learning and using heuristic percepts for improved problem-solving capacity. Herein lies a central point of departure for the investigations to be stimulated in this Project.

### Selected Investigations Emphasizing Heuristical Methods<sup>1</sup>

Recently several investigations of mathematical problem solving have attempted to teach explicitly heuristical precepts or with heuristical methods or to analyze problem-solving protocols using systems based on Polya's ideas. A number of studies have used various treatments of task-specific and general "heuristics" in attempting to improve subjects' problem-solving competence through instruction. Ashton (1962) gave ten weeks of heuristic-oriented instruction based on Polya's work to ninth grade algebra students. These students, when compared with a control group receiving conventional instruction, were better able to solve word problems.

Covington and Crutchfield (1965) also obtained superiority of the instructed children in measures of divergent thinking, originality, and perceived value of problem solving. These subjects used comic book format to engage students in developing "heuristics" for non-mathematical problems. Olton (1967) conducted an extensive test of the revised version of this program and confirmed the positive effects on a diverse set of performance indicators. His students achieved up to 50 percent higher scores on post-test measures where the teacher discussed each lesson, a finding which seems to support the value Polya has assigned to a "looking back" phase in his "planning heuristic." Jerman (1971) used The Productive Thinking Program and a Modified Wanted-Given Program (after Wilson, 1967) with fifth grade students and concluded that teaching problem solving in mathematics to students of this age can best be done in a mathematical context using a wanted-given approach, whereas either systematic approach to problem solving was more effective than not providing any systematic instruction.

Wilson (1967) and Smith (1973) compared subjects taught mathematical problem solving using either "task-specific heuristics" or "general heuristics" via self-instructional booklets. In each study the dependent variables were time measures and number of correct steps for each section of the criterion test. Both investigators hypothesized that task-specific heuristical instruction would lead to superior performances on the training tasks but poorer performances on the transfer tasks than would instruction in the use of general heuristics. Analysis of performances of Wilson's subjects on the training tasks and five transfer tasks revealed that only the general (planning) heuristic was superior to the others (a task-specific heuristic applicable only to the topic under study and a means-end heuristic); it is suggested that general heuristics learned in the first training task were practiced on the second task, thereby facilitating transfer. Thus, Wilson failed to confirm his central hypothesis and, in fact, found that one of his general heuristics (the

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<sup>1</sup>The reader is urged to examine other more comprehensive reviews of problem-solving research, including Kilpatrick (1969), Riedesel (1969), Simon and Newell (1971), and Suydam (1972).



planning heuristic) led to better performance on a training task than did the heuristic specific to that task. One possible explanation lies in the potential power of a heuristical maxim: the availability of the maxim to the problem solver and his efficiency in using the advice. In part, the problem solver must recognize that general heuristics are indeed general and therefore possibly applicable to solving an unfamiliar problem wherein known task-specific strategies are not useful. The limited learning time (two 3-hour sessions) and number of problem-solving episodes (about 20 problems) may have resulted in less practicing of general heuristics than may be necessary for subjects to become operable.

Smith's (1973) study, patterned after Wilson's, attempted to emphasize the generality of its general heuristics to a greater extent than Wilson did with his. Subjects were also given more opportunity to practice similar heuristic procedures in a variety of settings. The data for Smith's subjects failed to support the hypothesis that instruction in heuristics differing in level of generality leads to differences in performance on transfer tasks. Questionnaires and interviews suggested that, while subjects claimed to have used the heuristics when completing a given learning task, almost no subjects attempted to use the general heuristic on the transfer tasks. Smith conjectured the possible reasons for the apparent abandonment of the planning heuristic on these unfamiliar tasks. Often the more general heuristic does not always suggest itself; the problem solver must often make a conscious effort to reach into his "bag of heuristics" when he is stymied. Besides knowing strategies which he believes will be of use, the subject must think of trying them when he gets stuck. Adequate practice and encouragement, couched in successful applications of heuristical advice, seem absolutely essential for true operationality of such strategies. In positing suggestions for further study, Smith notes:

Investigators of human problem solving are probably well advised to incorporate some means of studying a subject's behavior in addition to examining test or time scores. Selected interviews or problem-solving questionnaires are one possibility. Some form of protocol analysis might provide valuable insights into the apparent lack of transfer power of general heuristic advice. In fact, exploratory studies might be more valuable than experimental ones, given the present state of our knowledge about human problem solving. A researcher who devotes his full energy to studying the problem-solving processes of his subjects rather than the products they produce may discover revealing behavior patterns. (pp. 100-1).

Kilpatrick (1967), using a system based on heuristical processes identified by Polya, analyzed problem-solving protocols of 56 junior high school students in relation to their performances on a battery of aptitude, achievement, and attitude scales. The processes used in

solving word problems seemed unrelated to observed systematic styles of approach to spatial and numerical tasks. Subjects who attempted to set up equations (pre-algebra students) were significantly superior to others on measures of quantitative ability, mathematics achievement, word fluency, general reasoning, logical reasoning, and a reflective conceptual tempo. Those subjects who used the most trial and error were higher than the others in quantitative ability and mathematics achievement. Those subjects who used the least trial and error had the most trouble with the word problems, spent the least time on them, and got the fewest number correct.

Lucas (1972) and Goldberg (1973) employed a modified version of Kilpatrick's (1967) system for coding and analyzing the problem-solving protocols of subjects following the application of instructional treatments designed to exhibit and use a general planning heuristic. Lucas made an exploratory study of the effects of teaching heuristic in calculus. Following an eight-week instructional period during which one class was taught using the style of teaching suggested by Polya along with prepared materials that defined and demonstrated the use of heuristical advice while another class learned calculus without explicit attention to heuristics, volunteers from both classes were interviewed individually and asked to solve problems while thinking aloud. The protocols of these thirty subjects were analyzed to identify strategies.

Lucas was able to isolate thirty-eight variables that represented heuristical ideas, indicators of difficulty, types of errors, and performance measures of time and product score. He found evidence of differences in performance between subjects in the heuristical and non-heuristical treatments, but his conclusions were stated tentatively, reflecting the purpose of identifying behavioral variables rather than hypothesis testing. Subjects from the heuristical treatment were judged superior in their ability to solve problems when the criterion was score on approach, on plan, or on result; they read the problems more easily and hesitated less as they worked. Some heuristics were used more frequently by subjects taught heuristically: working backward, using the result or method of a related problem, and introducing suitable notation. Heuristical training did not appear to affect the use of diagrams, the use of trial and error, the number of errors, the number of aborted solutions, or the ability to write proofs. Lucas concluded that Kilpatrick's system can be used to characterize college students' problem-solving behavior, and that heuristical maxims can be taught in calculus without infringing on course content. That heuristical instruction was no more time consuming than the kind of instruction given to the control class may be explained because the written instruction in heuristics was reserved for extra-class time. Larsen (1960) had found that calculus students instructed in heuristical ideas learned them, but apparently at the expense of normal course content.

Goldberg (1973) examined the effects of instruction in heuristical advice for writing proofs, suggested by Polya, on the ability of college students not majoring in mathematics to construct proofs in number theory. Two sets of seven self-instructional booklets written by the

investigator were used with 238 subjects enrolled in nine classes. Classes were randomly assigned to one of three different treatments: reinforced heuristic, unreinforced heuristic, and non-heuristic. Following the six-week experimental period, measures of understanding of number theory concepts, ability to construct proofs, attitude toward the self-instructional booklets, and attitude toward problem solving were obtained. The analyses indicated that heuristic instruction with reinforcement in class is relatively more beneficial than unreinforced or non-heuristic instruction with respect to understanding number theory concepts and writing proofs for high ability students. This result was also found favoring unreinforced heuristic instruction over instruction by imitating examples (non-heuristic). The non-heuristic self-instructional booklets were found more helpful, easier, and generally more appealing than the heuristic self-instructional booklets. Goldberg suggests that the non-heuristic materials were less threatening and more fun due to the inclusion of puzzles and number tricks whose purpose was to help to make the time to complete the booklets more comparable to the heuristical treatment.

A more important effect was observed in the more positive attitudes toward the problem-solving process of the subjects in the reinforced heuristic treatment than students whose instruction in heuristics was not reinforced in class. Applying a coding system to the written proofs of students scoring in the top third of the proofs posttest, she found that students given reinforced heuristic instruction used heuristics more than students who had studied number theory by imitating examples. Among the precepts noted, these students wrote "given" or "prove," rephrased the conclusion of the problem, used theorems more frequently than definitions, introduced suitable notation, and worked backwards in their proofs more frequently than proficient proof writers who were not given reinforced heuristic instruction.

Vos (1976) compared the effects of three instructional strategies on problem-solving behaviors in secondary school mathematics. Five particular behaviors (drawing a diagram, approximating and verifying, constructing an algebraic equation, classifying data, and constructing a chart) were identified as problem-solving behaviors or "heuristics." Each of three experimental treatments, classified as Repetition (R), List (L), and Behavior Instruction (B), involved an exposure to twenty mathematical problems but with variations in the placement of, and emphasis on, one of the five implied problem-solving behaviors. Simply stated, the treatments were: R presented the problem task only; L presented the problem task which was momentarily interrupted with a checklist of desirable problem-solving behaviors and individual written instruction in a specific useful solving behavior followed by a return to the problem task; and B initially presented individual written instruction in a specific problem-solving behavior followed by the same training problem task used in L and R. Each problem task administered through self-instructional materials took about twenty minutes to complete. Subjects in six mathematics classes (grades 9, 10, 11) at a

private Iowa school were randomly assigned within classes to one of the three treatments which occurred over about fifteen weeks. Post-treatment data included scores from a Problem-Solving Approach Test (PSAT) and a Problem-Solving Test (PST). PSAT consisted of two parts each having problem statements with choices indicating an approach that could best be used to solve the problem. Part I offered choices directly related to the five instructed problem-solving behaviors whereas Part II sought to measure transfer in using other problem-solving behaviors. The PST consisted of seven problems seeking a free written response with encouragement to write all their thoughts about the situation. In summarizing the results found from the various data analyses, Vos asserts that specific instruction in utilizing problem-solving behaviors increased the effective use of the behaviors. The evidence supports the idea that classroom mathematics instruction should involve specific instruction in a set of problem-solving behaviors.

Webb (1975) studied the problem-solving processes used by forty second year high school algebra students while solving eight problems. The aims of his study were (a) to consider how cognitive and affective variables and the use of heuristical strategies are related to each other and to the ability of high school students to solve problems, (b) to identify problem-specific strategies from those used in solving problems in general, and (c) to identify problem-solving modes of groups of students. Protocols from individual interviews were analyzed using a coding system adapted from Kilpatrick (1967) and Lucas (1972). Data from sixteen cognitive and affective pretests, frequencies of problem-solving processes, and total problem inventory scores were analyzed using principal component, regression and cluster analyses.

Webb observed that mathematics achievement was the variable with the highest relation to mathematical problem-solving ability. The use of heuristical strategies had some relation to mathematical problem-solving ability not accounted for by mathematics achievement. In particular, the component Pictorial Representation accounted for a significant amount of the variance. Thus, the processes used by students in this study added to their ability to solve problems beyond their mathematical conceptual knowledge (mathematics achievement).

Furthermore, Webb noted that students who used a wide range of heuristical strategies, on the average, were better problem solvers. Most of these heuristical strategies were found to be problem-specific. This implies that in order to solve several different problems, a range of problem-specific strategies needs to be employed. Strategies such as "specialization" and "successive approximations" were used at least once on six of the eight problems, but were used by more students on one or two of the problems. One possible reason for the restricted use of these strategies is that students only used the strategies in obvious ways and did not employ the strategies where they could be strategically used. Webb suggests that one direction for research would be to examine whether students can be taught to use such strategies to solve a wide range of problems.

Particular combinations of strategies appeared to relate to performance. Students who used a moderate amount of trial-and-error and a moderate amount of equations or who use equations often and trial-and-error seldom performed about the same on the Problem-Solving Inventory. Students who used a high frequency of trial-and-error and had a low use of equations did not do as well. These results are somewhat counter to Kilpatrick's (1967) observations of the relative effectiveness of trial-and-error methods. However, for these high school students and for the problems in the inventory, it appeared that trial-and-error as an approach to problem solving had a value as a supplementary process to the use of equations, but not as a replacement for the use of equations.

Kantowski (1974) conducted a "teaching experiment" (quite in the Soviet sense) as a clinical exploratory study of processes used by eight grade nine subjects in solving non-routine "to show" problems in geometry. She noted Talyzina's (1970) observation that subjects who were successful problem solvers in geometry introduced an "operative proposition" ("heuristic") into the solution. Kantowski observed that if the "heuristics" used by her students were goal-oriented (that is, specifically related to the conclusions of the problem) the solution tended to be more efficient. She observed a dramatic increase in the use of goal-oriented "heuristics" across her instructional treatments on "heuristics" and geometry problem solving.

But she made an even more penetrating observation. Of course, valid deductions are often essential to solving mathematics problems. Such deductions are commonly made by analysis and synthesis. She took analysis to be what Polya refers to as "decomposing" or making inferences from what is known. Synthesis, on the other hand, is a "recombining" of facts to form a new whole. She examined where these analytic and synthetic deductions occurred in the sequence of processes during solution and their relationship to the use of "heuristics." The manifestation of regular patterns of analysis and synthesis among successful problem solvers is striking. In a high percentage of cases these regular patterns of analysis and synthesis were immediately preceded by a goal-oriented "heuristic." In cases where non-goal-oriented "heuristics" were introduced, the patterns of analysis and synthesis were generally irregular and superfluous.

McClintock (1975) reported the relative effects of verbalization of "heuristics" on transfer of learning. In three one-hour periods of instruction (methods were discovery, expository or control) generalizations for the sums, sums of squares, and sums of cubes of the first  $n$  natural numbers were taught (modes were teacher verbal instruction, self-instructional reading, or combined teacher verbal and self-instructional reading). Following initial learning students attempted to two-item practice test whereupon a "heuristics verbalization" group responded verbally to the request to state what they recalled having done in solving the problems with encouragement to use Polya's four

phases for their description. Thereafter, all groups were given transfer tasks (12 problems to solve) related to the taught generalizations. Significant effects for method of instruction, "heuristics verbalization," and interactions were found. It appeared that a combined instructional mode followed by the "looking back" of the verbalization of "heuristics" tended to produce greater transfer to problem-solving tasks.

### "Looking Back" at These Studies

What do these few premier investigations offer as results or directions for future research? What features seem common among these studies? Are there research methodologies, variables or designs appearing to encompass more salient aspects of the behaviors we should or might be studying?

Defining constructs. The plague of ill-defined constructs which permeates much of educational research is manifested in most of these investigations. The broad implications of the notion heuristic result in its usage being at best varied but more often unclear. In one sense it involves the "science" (or art) of studying and describing the mental processes or operations of solving problems. This would seem to connote an applied epistemology: The study of, and use of, knowledge itself. Despite its focus on mathematical problem solving serving to provide considerable clarification, Polya's more practical approach does not relay a cognitive psychological theory of the nature and usage of heuristic.

The studies described earlier generally use the term "heuristics" (plural) in the sense suggested by Kilpatrick: ". . .as any device technique, rule of thumb, etc., that improves problem-solving performance" (1967, p. 19). The maxims, questions, precepts, ploys, and suggestions offered by Polya and others are most often referred to as "heuristics." Usually no more than implicitly are the actual behaviors of the problem-solving act considered. Particularly missing are discussions of the mental operations and cognitive structures which are (a) necessary to assimilate the heuristical precept as a potentially powerful item of knowledge for future problem solving, (b) required to recall and then apply the heuristical advice, (c) "triggered" by the recall or suggestion of an heuristical statement.

Finally there is further confusion resulting from the use of heuristical notions in conceptualizing instructional methods for teaching about the maxims and their use in problem solving. It would seem that one could teach heuristical advice as generalizations via expository instruction. Yet Polya implores the teacher to use a pedagogy of problem setting, teacher questioning, active learner participation and choice, and post-solution discussions that reflect the techniques of an heuristical problem-solving approach.

Another notion requiring further clarification is problem-solving strategy. Various references can be found to a "trial and error strategy," an "inductive strategy," "heuristic strategy," "information processing strategy," an "algorithmic strategy," or a "strategy of indirect proof." The considerable range of interpretation occurring in these examples illustrates the confusion. Yet the intuitive idea of strategy has appeal as a useful idea for describing or contrasting approaches to a problem solution.

Studying processes. Surprisingly, few of these studies used as dependent measures the actual process sequence trace of the subject's problem-solving act. Rather time to solution, number of correct steps, and correct problem answers are used. Kilpatrick, Lucas, Kantowski, and Webb interviewed subjects to obtain protocol analyses of the process sequences. Yet even here the primary dependent measures became type and frequency of occurrence of various heuristical precepts. Only Kantowski reports identified patterns in the sequences noted for her subjects.

The particular manifestation of student behaviors, including explicit as well as apparent use of heuristical ploys, may be a way to characterize solution strategies. Similarities across problems (both in content and sequence) of processes used by a subject could be described as a solution strategy known to that problem solver. Or consistent displays of process sequences across subjects for a particular problem or type of problem could empirically describe the solution strategies commonly associated with that problem.

The protocol analyses undertaken in some of these studies engaged the seminal coding system, or variations thereof, provided by Kilpatrick (1967). Obviously the nature and quality of process analyses will be contingent upon the sensitivity and comprehensiveness of the interview and the template of the coding scheme subsequently used on the protocols.

Generality of heuristics. Several of these investigations dealt with a "generality of heuristics" dimension. Recall Polya's recognition of the generality characteristic of his questions and suggestions. As a teacher, he seeks to offer advice as unobtrusively as possible so that steps in the emerging solution path do not become too obvious to the solver. At the same time he wants the advice to be widely applicable to many different problems. Any research findings about "general vs. task-specific" heuristics are very tentative. Most investigations were able to observe positive effects from subjects being taught or using more general heuristical precepts. Interestingly, few of these studies offer direct evidence regarding whether the subjects indeed learned the taught "heuristics." Most studies employed transfer tasks to observe knowledge of heuristical advice by observing its use in problem solving.

Instructional intervention. Following the artistic lead offered by Polya, many of these investigators have sought to improve the problem solving performances of their subjects by directly providing special instruction in "heuristics." Programmed instruction booklets were used in several studies. Some investigators incorporated especial concern for accompanying such booklets with teacher reinforcement (discussion) of the "heuristics" employed. Yet the results of the effects of explicitly developing a "looking back" phase in the instruction are mixed.

The characteristics of the instruction actually offered are not clearly detailed in reports of some of these investigations. Among the variants one finds (1) the extent to which certain heuristical maxims are specifically taught, (2) the manner in which maxims are isolated and illustrated as the single or primary tactic in a problem solution, and (3) the degree to which an "heuristic" question or advice is explicitly stated as opposed to being modeled but never "pointed at."

Notable results. Both conclusion-oriented and exploratory investigations are represented in the studies reviewed here. Obviously caution must be exercised in accepting the results. Yet certain findings appear to be evident across several of these studies:

1. A student's background knowledge of mathematics appears to be a dominant factor in successful mathematical problem-solving performances. This observation supports the importance of carefully building-up the problem solver's knowledge of mathematical ideas. However, the relationships among instructional variables and problem-solving competence are not clear from this research. In particular, it is unclear what effects explicit "instruction in heuristical methods" may have upon knowledge structures, especially within problem-solving tasks.
2. Students given special treatments which feature problematic tasks and solving processes are often rated as relatively better or improved problem solvers. Thus, solving problems and attending to solving methods do appear to obtain positive results. Yet, beyond this global maxim, "solve problems and reflect upon your solutions," considerably more detailed information is needed to guide mathematics teachers and students. Some potential directions are suggested by the instructional and task variables included in these investigations.
3. Certain heuristical plays or maxims appear from these studies to be more commonly taught or used. These included trial and error, successive approximation, working backwards, drawing a pattern or representation, and inductive pattern searching. Perhaps these are more immediately used



because they are among the solving tactics used by humans in coping with all manner of task situations. They are deeply habituated and, therefore, more naturally and automatically called upon. As Polya noted, the qualities of generality and common sense may be significant in obtaining heuristical competence.

4. The influence of idiosyncratic traits which a student brings to a problem-solving episode is unclear. Few of these investigations attempted to study subject variables. Yet it seems crucial to effecting improved mathematical problem solving to be clear about the roles played by a student's aptitudes, preferences, cognitive structures, memory, learning styles, or personality. Furthermore, it may be important to discern individual trait (i.e., relatively stable and long-term) and state (or situational) factors in problem solving.

#### "Looking Ahead": Toward Coordinated Research of Mathematical Problem Solving

One of the unique features of a research consortium ought to be the manifestation of a more concerted thrust on the problems under consideration than the individual investigators, working separately, might produce. Put another way, the "whole should somehow become greater than the simple sum of all parts." Coordinated team research in an area as complex as the learning and teaching of mathematical problem solving will not be easy. Yet perhaps the conditions for creating focussed research efforts and results are now only becoming existent in mathematical education in this country.

This section examines certain points of view regarding prospective research in the Problem-Solving Project. These ideas are expressed to engender discussion both during and following the Research Workshop. It must be clear that all committed participants in the intellectual consortia connoted by the GCSLTM must shape its eventual contributions. And this will certainly be a long-term effort. To execute change, real change, in the teaching and learning of mathematical problem solving in our schools will require no less.

The choice of emphasis in "instruction in heuristical methods" will cast a certain direction to the Problem-Solving Project. Inherent in this choice are several hypotheses. Heuristical methods of problem solving:

- a. can be learned,
- b. can be taught,

- c. if effectively used, do improve problem-solving performances, and
- d. along with its pedagogical counterpart, heuristical methods of teaching, can become a viable part of mathematical curricula.

In a global sense it could become the overriding mission of the Problem-Solving Project to study and/or test these general hypotheses or conjectures. At the same time the focus on "instruction in heuristical methods" must not inordinately constrain our research. While having a definite "applied" (i.e., classroom-oriented) characteristic, we must also recognize and encourage more basic, theory-oriented efforts. The pattern of the recent Soviet research, incorporating both aspects, may be a useful paradigm to follow.

### Directions for Future Research

Several general features of investigations to be encouraged in our consortium are next proposed. First, the processes, per se, of mathematical problem solving must be studied. Most past mathematics education research which considered problem-solving outcomes has examined various treatments and dependent measures that ignore the actual processes used by the subjects during their problem-solving acts. The solution (i.e., final "answer" or proof) of a mathematics problem, however lucidly set down, is typically an inadequate trace of the processes used to arrive at that solution object. Ample direction for studying the cognitive processes used by students in problem solving can be found in the works of Brownell (1942), Duncker (1945), Buswell (1956), Wertheimer (1959), Polya (1962), Kilpatrick (1969), and Kantowski (1974). Of particular interest is the emphasis taken by a number of Soviet researchers in studying the dynamics of mental activity during mathematical problem solving (Kilpatrick and Wirsup, 1969).

Secondly, the problems used with subjects should be non-trivial mathematical problems of the sort they might meet in the classroom. By a mathematical problem is meant a challenge encountered in a task environment, which is itself perhaps only partially known to the subject, wherein the concepts, relations, operations, transformational procedures, and models of mathematics provide the major elements or vehicles for solving the challenge. Laboratory studies of the psychologist have rarely dealt directly with the complex behavior appropriate for solving a challenging mathematics problem. Many reasons can be noted for selecting simple tasks, such as card sorting or level pulling. At the same time mathematics educators have generally hesitated to apply any conclusions stemming from such research because the tasks have been unrelated to the type of mathematical problems posed by the mathematics teacher. To assure relevance to mathematical education, we should emphasize commonly used as well as nonroutine settings which utilize appropriate mathematical concepts, principles, and skills either known or readily learned by the subjects.

The research methodology should make major use of qualitative methods, small groups of subjects, and long-term genetic approaches to study the learning and development of problem-solving competence. There may be a growing sense of the need for the careful conduct of clinical investigations in mathematics education research. Brownell noted promising changes in psychological research on problem solving, including the setting of problems which "mean" something to the subject, concentrating attention on not merely the errors and successes but on the way the subject proceeds to attack and solve the problem, and attaching greater importance "to qualitative descriptions of significant behavior to supplement or to replace purely quantitative descriptions" (Brownell, 1942, p. 419). Certainly the influence of Piaget and his followers in demonstrating the efficacy of such methodologies has been great. The recent appearance of the series, Soviet Studies in the Psychology of Learning and Teaching Mathematics (Kilpatrick and Wirszup, 1969) has generated further interest in approaches aimed at penetrating into the child's thoughts to analyze his mental processes. Menchinskaya (1969) described various methodologies used within the genetic approach, including the "teaching experiment" in which study is combined with pedagogical influence and entire classes of children are involved over a number of years in order that more valid judgments about the changes that occur in mental activity as a result of instruction might be made. Kilpatrick advocates this methodology by suggesting that the researcher "who chooses to investigate problem solving in mathematics is probably best advised to undertake clinical studies of individual subjects. . .because our ignorance in this area demands clinical studies as precursors to larger efforts" (Kilpatrick, 1969, p. 532).

Wilson (1973) discussed the role, features, and credibility of clinical intervention research. Three major purposes research must fulfill were identified: generating hypotheses with antecedent probabilities confirming hypotheses, and constructing guiding models and explanatory theories. Theory construction was assigned to a class referred to as Analytic-Synthetic Research while the confirming of hypotheses was assigned to Experimental Research. Normative Research deals with activities designed to generate hypotheses concerning facts and those connections between facts which exist in nature. Clinical Intervention Research, also conducted from the generative purpose, is aimed at producing hypotheses about the connections among facts which might be brought into natural existence by some intervention. Among the distinguishing features of Clinical Intervention Research, Wilson noted the methods of data collection (emphasizing interviews), the type of data collected (primarily qualitative), the analyses performed on these data (often categorical), the selection of subjects (typically non-random) the number of subjects (usually only a small group), the length of time for subject involvement (extended periods), the nature of the treatments (loosely pre-planned but dynamically modified for individual subjects), and the contents of reports (extensive, systematic descriptions of treatments and apparent effects coupled with a conscious, guided search for patterns among the idiosyncratic performances that can be translated into testable hypotheses).

A research methodology which seeks to elicit information about the subjects' cognitive processes must utilize carefully conceptualized task environments wherein the subject can operate naturally, openly, and productively. Our tasks must embody the potential for stimulating cognition either directly or isomorphically characterized as mathematical thought. Within such task environments our subjects would operate with other participants or, at times, with only the experimenter. The productive problem solving of children working together toward a common goal will have direct importance for classroom applications of this research. Although we can recognize the inadequacies and potential distortions, we should explore probing techniques to encourage the subject to give a verbal self-report of what he is or has been thinking, thereby prompting a phenomenological, stream-of-consciousness record of mental processing. Thus, to summarize, in the current state of knowledge with respect to mathematical problem-solving behavior, primary emphasis must now be given to designing appropriate task environments, carefully observing over a long period the individual child's spontaneous and learned problem-solving behavior, reporting detailed case studies directed at portraying the child's learning and development of problem-solving competence, and generating hypotheses, procedures, measuring instruments, and designs for future experimental investigations.

The primary emphasis in our immediate research should be on conducting "teaching experiments" (in the Soviet sense) to obtain further detailed information on mathematical problem-solving heuristics as teachable-learnable-transferable knowledges. These investigations would likely have school-based, yet clinical, treatments which: (a) explicitly teach heuristical precepts within mathematical problem solving, (b) use "heuristic teaching" methods, and (c) study the cognitive prerequisites for, and mental processes of, acquiring and using such general precepts in mathematical problem solving. Results of such investigations should lead to empirically derived "maps" of the domains to be considered in learning and teaching problem solving. Careful delineations of taxonomies or typologies of problems, "heuristics" or "strategies," and mathematical knowledge structures need to be formulated.

At the same time, it seems of paramount importance to sponsor and encourage analytical research. Little or no substantive theory-building has been done relative to the teaching, learning, or using of heuristical methods in mathematical problem solving. In recent years there has been a notable increase in the interest of mathematics education researchers for studying the use of heuristical advice in problem solving. Yet these scattered investigations have not been predicated upon, nor led to, any but the narrowest of theoretical bases. The assumptions upon which an instructional emphasis in heuristical methods is based must be better explicated. Those variables that may be accounting for productive problem-solving performances, learner difficulties in problem solving, and effective teaching and

modeling of problem solving must also be identified and defined. In short, fundamental questions need to be systematically generated and studied. Which heuristical maxims are "teachable objects"? What teaching "moves" or strategies might foster the acquisition and use of heuristic? What is the nature of a child's learning with respect to heuristic? To what extent might a learner's stage of cognitive development account for the ease or difficulty in acquiring or using heuristical methods during problem solving? Are there useful taxonomies or typologies of mathematical problems for illustrating various heuristical maxims? These types of questions demand extensive analyses as well as sensitive empirical treatment.

The interface of available theoretical formulations in cognitive development psychology and in teaching strategies with our heuristical emphases must be considered. The proposed emphasis in "instruction in heuristic methods" necessitates concerns for teaching factors. The model upon which the Teaching Strategies Project of the GCSLTM is based does not include an analysis of "moves" in teaching problems, in teaching about problem solving, or more specifically in teaching heuristical maxims (though the latter might well be construed to be principles or generalizations within the present model). Yet the power and generalizability of this teaching strategies model suggests that its fundamental features may be useful in building a similar model for teaching heuristical methods and problem solving. Such model building and subsequent model testing ought to be a central feature of our future work. The production of a potentially huge bank of recorded lessons or episodes from our "teaching experiments" would allow an easy access to a variety of excellent teaching exhibits to be analyzed in such model building.

The relationships between cognitive developmental psychology and the development of mathematical problem-solving competence must be studied. In particular, aspects of Piaget's concrete and formal operational thought should be examined for possible connection with the learning and use of heuristical methods. Conscious use of heuristics in problem solving would seem to involve thought which encompasses multiple operations, combinatorial processes, isolation of task variables, logical operations, flexibility (e.g., reversibility-reciprocity) or other aspects of Piaget's theory. We ought to be able to teach the heuristical questions as items of knowledge. But can we teach all subjects to selectively use such heuristical advice? Polya noted the importance of the mental operations implied by an heuristic precept. We need to map out in finer detail how mental operations and structures may describe contingencies for successful problem solving. That is, a more penetrating analysis is needed of the cognitive operations essential to allow the mind to consciously employ executive control over, and choice among, the problem-solving strategies known to the student. In particular, caution must be exercised that the operativity we wish to see exhibited in the application of heuristical methods of problem solving does not become mechanical rule-use by students who have been unable to assimilate such knowledge.

Perhaps an equally viable question would pursue the apparent effects of heuristical knowledge on cognitive operativity. To be sure, many of the task environments used by Piagetians are problem-solving ventures. A subject's spontaneity in approaching and dealing with the task is often crucial to ascertaining the stage of operativity. Yet heuristical knowledge would conceivably influence the fashion in which a subject encoded and operated on a task. For example, would subjects who knew how to approach problems with quite well-organized, inductive pattern searching strategies and who did in fact solve some of the classical formal operations tasks (e.g., pendulum, balance beam, hidden magnet) as an apparent result, have essentially been accelerated in their development?

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The Teaching Experiment and  
Soviet Studies of Problem Solving

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The purpose of this paper is threefold; first, to examine the typically Soviet research methodology known as the teaching experiment; next, to review several of the Soviet Studies related to mathematical problem solving in which some form of the teaching experiment was used; and, finally, to reflect on ways in which aspects of this methodology could be applied in this country in research dealing with the processes involved in mathematical problem solving.

Rationale for a New Methodology

A perspicacious grasp of the Soviet concept of the "teaching experiment" requires a thorough understanding of the forces that led to its conception and some reflection on the rationale for its development.

Among the primary forces that necessitated the evolution of a new research methodology in the U.S.S.R. was the influence of the philosophy of the collective in the post-revolutionary society. The Soviet attitude toward learning and instruction was a strong reaction to the concept of the class system of pre-revolutionary Russia. In the spirit of the theory of dialectical materialism the Soviets assumed instruction, not native ability, to be the major factor in intellectual achievement. They believed that except for cases involving organic damage or severe retardation, all children have the same potential for academic accomplishment.<sup>1</sup>

Whereas learning theories and pedagogical principles had previously been based on experimental and theoretical psychology, the Marxist perception of the dynamic interrelationship between pedagogy and psychology emphasized the influence of instruction and content on psychological

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<sup>1</sup>Only a minority of pedagogical researchers led by V. A. Krutetskii are somewhat dissonant with this main stream of thought. Krutetskii, whose research will be discussed in a later section of this paper, is investigating variation in ability.

growth. This point of view compelled a search for a research methodology, different from the "cross-sectional" type of investigation, that would permit researchers to observe qualitative effects of various forms of instruction. Such a direction obviously mandated the organization of research as well as techniques that would allow researchers not merely to observe complex processes involved in learning such content as reading, grammar, and mathematics, but that would, in fact, influence the development of these processes. The new research methods would have to include longitudinal observation and evaluation; they would have to permit a researcher to study changes in mental activity as well as the effects of planned instruction on such activity.

To this end, pedagogical and psychological research were tied together in their organization under the Academy of Pedagogical Sciences (Reitman, 1962). Since the primary value of psychological research under the Soviet regime was seen to be the improvement of instruction, Academy studies of thinking necessarily involved the teaching methods most effective in producing learning and independent thinking, and, conversely, studies related to instruction dealt with complex mental processes.

### The Influence of Vygotsky

The "teaching experiment" grew out of the "individual psychological experiment" introduced in the twenties by Lev Semyonovich Vygotsky, the psychologist-educator who left an indelible mark on Soviet pedagogical research before his early death in 1934.

In the Marxist tradition, Vygotsky asserted that specifically human mental processes are not inborn but formed, and that their development is totally dependent on how they are taught. He characterized intellectual development as evolutionary or shaped by adaptation to external environment and not embryonic which he interpreted as development flowing more or less smoothly according to a stereotype. According to Vygotsky, all mental processes occur only by acquisition as a result of internalizations after some external activity (El'konin, 1967; Gal'perin, 1967). Moreover, clinical data collected in early studies convinced him that the development of certain mental processes was accompanied by changes in cognitive structure at various levels of sophistication of function of these processes (Kostyuk, 1968). He found, for example, that the processes of analysis, synthesis, comparison, and generalization exhibited definite levels, and noted positive effects of various pedagogical practices on these levels.

These factors in the revolutionary concept of the formation of mental processes, the indication of changes in cognitive structure with mental growth in certain operations and this conviction of the primacy of instruction in mental development led to Vygotsky's conception of a genetic instructional research methodology that would focus in the qualitative aspects of thinking and learning. He conceived of an "instructional experiment" that would be a systematic reproduction of

processes as they develop under various instructional procedures. In his "laboratory," often the school setting, he tried to follow the course of development by "experimentally evoking the genesis of voluntary attention" (El'konin, 1967). Because the processes were observed only periodically, Vygotsky attached great importance to his concept of the "zone of proximal development" (Gal'perin, 1967; Kostyuk, 1968), where the passage from lack of knowledge or lack of ability to operate to possession of knowledge of operational ability and the corresponding change in cognitive structure could be observed.

Perhaps artificially, Vygotsky distinguished between "simple" and "scientific" concepts, a difference not in the content of concepts but in the way concepts are mastered (El'konin, 1967; Menchinskaya, 1969b; Talyzina, 1962). He saw the former as learned spontaneously and inefficiently from "object to definition" through daily experience while the latter were learned from "definition to object" through carefully planned instruction. This distinction, which further emphasizes the Soviet view of the primacy of instruction was seen as necessary to Vygotsky since he felt that in the acquisition of "scientific" concepts and relationship between instruction and development was most clear and capable of most complete investigation. It was with the development of "scientific" concepts that Vygotsky's studies were concerned.<sup>2</sup>

The method introduced by Vygotsky was, in a sense, modeling rather than empirically studying the processes as they developed and, studying the results of the learned behavior in a clinical setting.

Vygotsky began using these genetic experimental techniques in studying the relationship between language and thought. His influence soon spread to other disciplines, and although his basic concept of the pedagogical experiment remained the same, it began to take on different forms to correspond to varying research needs.

### Characteristics of the "Teaching Experiment"

If one word had to be chosen to characterize the "teaching experiment," it would most likely be the term "dynamic" since it is movement that interests the Soviet researchers--movement from ignorance to knowledge, from one level of operation to another, from a problem to a solution. The aim of this research is to "catch" processes in their development and to determine how instruction can optimally influence these processes. Unlike most experimental designs used in this country,

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<sup>2</sup>Recently, Menchinskaya (1969b) and Talyzina (1962) have taken issue with Vygotsky's assumption that concepts are best learned "scientifically." Studies related to activity learning (Zykova, 1969; Kalmykova, 1962) support their objections.

the "teaching experiment" cannot be completely characterized by describing sampling procedures, experimental groups and test statistics. The label is actually a generic term for a variety of pedagogical research forms in which the strictly statistical analysis of quantitative data is of less concern than the daily subjective analysis of qualitative data. Most studies deal with some aspect of the formal school situation although the data are often gathered only from a sampling of "strong" "average" or "weak" students who are generally categorized and selected with the aid of the classroom teacher. The data collected are often qualitative, obtained in a clinical setting by recording verbal protocols for future analysis. Underlying this procedure is one of the salient features of the "teaching experiment," its compensatory nature. The quantity of macroscopic data (such as objective test scores) generally acquired in an experimental study is exchanged for microscopic detail of processes observed using a small sample. Probing interviews and exchanges with individual students add to any group data collected to support generalizations resulting in decisions for future instructional sequences.

Other general characteristics of the "teaching experiment" include its longitudinal nature (instructional treatment is applied and data are gathered over an extended period), the planning of instruction in the light of observations made during the previous session, and extensive co-operation among classroom teachers and researchers. It is procedurally acceptable to give hints to the subjects during testing, so that any learning in the testing situation may also be observed. In most cases results are reported in the form of a narrative that includes an analysis of observed behaviors and conclusions drawn from the analysis. Any quantitative data are generally reported using descriptive statistics. Inferential test statistics are seldom used.

### Some "Teaching Experiments"

Menchinskaya broadly defines the "teaching experiment" as "study combined with pedagogical influence" (1969a). She describes two forms of this research widely in use (Menchinskaya, 1969b). The first is the "experiencing" form in which only one mode of instruction is employed and observations are made in a clinical setting to determine its influence on mental processes. No explicit comparison is made to any other instructional procedure.

The Gal'perin and Georgiev study (1969) and the follow up study by El'konin (1961) are examples of research using the "experiencing" form. Both involved the introduction of mathematical concepts using the unit of measurement instead of the concept of number. In the El'konin study, although no explicit comparisons are made, the subjects chosen were judged to be very low in mathematical concepts initially. Thus, the fact that this group learned all mathematical concepts and skills required at their grade level implied a judgment regarding the value of the method. The study employed the latest form in the evolution of the "teaching experiment" and one that should be of

interest to American researchers who are attempting to bridge the gap between research and what actually occurs in the classroom. The experimental classroom used was equipped with one-way glass and with a TV camera lens mounted in the classroom and connected to a screen in the adjoining laboratory, which was also equipped with booths for individual experiments. All group and individual sessions were recorded on tape for future analysis. Although the general course outline and content to be covered were determined in advance, the experimenters, teachers and aides observed the class sessions held during the day, discussed the lessons and planned activities and instruction for the following session on the basis of what occurred each day. The individual experiments highlight another feature of the "teaching experiment," that of probing for hunches on which to base new instructional strategies. Again, the dynamic nature of the "teaching experiment" is evident here as the experimenters attempt to capture processes as they are being formed and to determine optimal strategies of instruction. This procedure may be useful as a preliminary to pilot testing to determine plausible hypothesis to be tested in future experimental studies.

The "experiencing" mode of the teaching experiment was also used by Krutetskii in his studies of mathematical abilities (1965, 1969, 1973). The investigations were conducted by Krutetskii and his students between the years 1959 and 1965 using students from the second through the tenth grades. By analyzing solutions of carefully organized sets of mathematical problems generated over periods of about two years with the same students, Krutetskii was able to delineate characteristics of students with high ability in mathematics. The organization of the problem presentation was instructional; as the problems were solved, mental processes were observed in their development. The problems included those requiring generalizations and algebraic proofs, and those with visual-graphic and oral-logical components, among others. Krutetskii emphasized that although some quantitative data were gathered (e.g., the number of problems solved and the time to solution), the dynamic indices, such as progress in qualitative aspects of problem solving, were more valuable than the static, quantitative ones. These are reflected in the components Krutetskii enumerated in the structure of mathematical abilities, namely (1) the formalized perception of mathematical material (2) quick and sweeping generalization of mathematical material (3) curtailment of thought (4) flexibility of thought (5) striving for economy and (6) a mathematical memory.<sup>3</sup> In a recent publication, Krutetskii took a definite stand on the existence of levels of ability (1973), a position opposed to the basic Soviet philosophy.

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<sup>3</sup> In his earlier writings Krutetskii included spatial skills in the structure, but later removed it from the "obligatory" structure.

The second form of the "teaching experiment" is the "testing" mode, one more closely related to our own experimental studies. This procedure was used by Kalmykova (1962) in a study related to mathematical applications in physics. Aspects of the research that clearly distinguishes it from our experimental studies include the type of data collected and the form of analysis of these data. With the help of the classroom teacher the subjects in the study were divided into "weak" and "strong." One half of each group was assigned to each of two methods of instruction--method "A" which was essentially expository and in which the teacher outlined the procedures to follow in completing the exercises, and method "B" which was essentially a heuristic teaching technique. A variable called "rate of learning" was determined by the number of problems needed in the instruction and the time necessary for a subject to complete exercises independently. Kalmykova states that there was no significant difference in the effects of two methods for the "strong" students. In spite of the fact that the average number of problems and the time required for mastery were not significantly different even in the case of the "weak" pupils she concludes, on the basis of analysis of clinically obtained data, that method "B" was superior for the "weak" students since they exhibited higher levels of analytic-synthetic activity in the solution of the control problems.

In another study Kalmykova (1975) uses a form that does not clearly fall into either the "experiencing" or the "testing" category but contains elements of both. In her initial research study on analysis and synthesis in problem solving, Kalmykova observed various teachers as they taught problem solving in elementary school classrooms and then examined the problem solving behaviors of their students. She analyzed in detail the instructional strategies of one particularly successful teacher, V.D. Petrova, and suggested elements that should be included in all problem solving instruction based on her analysis of Petrova's techniques. Kalmykova herself then applied these techniques with some success in instructing a group of "weak" students. Petrova's method of instruction, which emphasized both analysis and synthesis in problem solving, was compared to the more structured "classical analysis" methods used by the other teachers. These comparisons used observations of problem solving behaviors of the students from all participating classes. Thus, both the "experiencing" form and the "testing" form were used to some extent by Kalmykova in the same study.

### The "Teaching Experiment" and Our Research

Any attempt to compare the Soviet methodology to research designs used in this country would be analogous to an attempt to answer the question: "Who is the better athlete Chris Evert or Olga Korbut?" Just as an athlete is judged on standards related to her event, each research methodology must be examined in the light of its purposes and the philosophy of education encompassing it. The Soviets are concerned with the qualitative aspects of mathematics learning and

problem solving. Thus far we in this country, with few exceptions, have focused on the quantitative. Perhaps the answer to effective problem solving research lies in a compromise--a merger of the two methodologies through studies that involve both aspects.

Paradoxically, the essential differences between the "teaching experiment" and research in this country account for what could here be considered the most severe limitations as well as the desirable strengths of the Soviet methodology. Because the instruction is often determined by what is found in preliminary analysis and the duration of the experiments is often a year or longer, this approach introduces variables that would be considered invalidating to most American researchers. The time between the initial testing and the introduction of the instruction could result in learning on the part of subjects who were identified as not having the desired skills initially. Because analysis is logical, the introduction of hints during testing is not considered undesirable in Soviet research. Experimental control in our sense gives way to the opportunity to "catch" the learning of a concept or strategy and to suggest instructional techniques to insure mastery. On the other hand, the instructional experiment allows the researcher to observe how a subject is operating and to determine levels of sophistication (for example, elegance in problem solving) instead of mere numbers of correct solutions. Such diagnostic techniques permit the discovery of erroneous concepts as well as "strokes of genius."

Few researchers in mathematics education in this country have concerned themselves with the detailed study of the development of processes in mathematics. Kantowski (1974) investigated processes used in the development of skills in solving geometry problems; Lucas (1972) and Goldberg (1973) studied processes related to problem solving in the Calculus and number theory, respectively.

Although process research in this country has been sparse, several recent studies related to problem solving could be modified to include a process component. For example, in the Wilson (1967) study involving the level of generality of heuristics used in instruction, a matrix sampling technique could be used and qualitative data collected from representative subjects in each of the cells. Analysis of such data could serve as a valuable supplement to the statistical analysis of the quantitative data by providing information on how the development of processes is affected by the level of generality of the heuristics used in instruction in each of the content areas, and perhaps by suggesting other instructional techniques.

Another possibility would be to randomly select individuals from intact classes such as those used in the Goldberg (1973) and Lucas (1972) studies and to follow identical instructional techniques with these individuals while gathering qualitative data from their verbal and written protocols along with the quantitative data from the remainder of the classes. Such observations made using subjects on various ability levels could suggest Aptitude-Treatment-Interaction studies for teaching problem solving. Other suggestions for studies related to process research may be found in Kantowski (1974).



The gathering of qualitative data is but one step in progress toward understanding the processes of mathematical thinking. Methods of analyzing the data are needed, and more importantly, methods for communicating the results to other researchers, and to classroom teachers must be explored. Finally, the ultimate goal is to find ways to use the data to improve classroom instruction and to positively affect mathematics learning and problem solving in the classroom.

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Mathematical Problem Solving In The Elementary School:  
Some Educational And Psychological Considerations\*

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Introduction

One of the most important goals of elementary school mathematics is to develop in each child an ability to solve problems. In recent years more and more emphasis has been placed on problem solving in the elementary mathematics curriculum. A cursory look at the scope and sequence charts of the most popular textbook series points out this trend. In each of these series problem solving is identified as one of the key strands around which the mathematics program is built. At the same time there is concern among teachers, mathematicians, and mathematics educators that these programs are doing a poor job of developing problem solving ability in children. Points of view which are representative of the dissatisfaction with current programs are found in the reports of the Snowmass Conference on the K-12 Curriculum and the Orono Conference on the National Middle School Mathematics Curriculum held during the summer of 1973. These reports called for extensive modification of current mathematics programs to include a more systematic approach to providing instruction in problem solving.

The current concern should raise a number of questions in the mind of anyone interested in the mathematics education of children. Exactly what is problem solving? Can students really be taught to be better solvers? If problem solving is so important and good problem solvers are not being developed, what steps should be taken to change present instructional practices? Certainly an answer to the first question must be obtained. So, before proceeding any further a definition of problem solving should be provided.

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### Definition of a Problem

A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution.

Any one of a number of other definitions of a problem would be satisfactory for the purposes of this paper (e.g., Bourne, Ekstrand & Dominowski, 1971; Davic, 1966; Henderson & Pingry, 1953; Newell & Simon, 1972). Let it suffice to say that any reference to a problem or problem solving refers to a situation in which previous experiences, knowledge, and intuition must be coordinated in an effort to determine an outcome of that situation for which a procedure for determining the outcome is not known. Thus, finding the length of the hypotenuse of a right triangle given the lengths of the two legs probably does not involve problem solving for the student who understands the Pythagorean Theorem, but may be problem solving of a complex nature for the student who has not been exposed to the Pythagorean Theorem.

Since problem solving is viewed as such an important part of learning mathematics, it seems natural to analyze carefully what is involved in the process so that effective instructional techniques can be developed. There is little or no argument on this point. Everyone agrees that serious attention must be given to instructional issues related to problem solving. However, beyond this point there is little, if any, unanimity of opinion concerning the process of problem solving.

Even the most successful problem solvers have difficulty in identifying why they are successful, and even the best mathematics teachers are hard pressed to pinpoint what it is that causes their students to become good problem solvers. Unfortunately, in spite of the volumes that have been devoted to problem solving what is now universally accepted knowledge about problem solving can be boiled down to George Polya's words of advice to mathematics students: "Use your head." (Professor Polya's final statement in a presentation at the 1974 annual meeting of the American Mathematical Society.)

Out of frustration over an inability to deal successfully with the problem solving dilemma, mathematics educators have turned to psychology for guidance. The nature of problem solving and the measurement of problem solving ability have been the objects of considerable attention by psychologists (representative reviews of psychological research in problem solving have been written by Bourne & Dominowski, 1972; Davis, 1966; Green, 1966). Typically, psychological reports of problem solving research begin with a statement like: "Research in human problem solving has a well-earned reputation for being the most chaotic of all identifiable categories of human learning" (Davis, 1966, p. 36). Indeed, it has only been during the last twenty-five years that a major point of view or technique has

developed which attempts to isolate the important variables which influence problem solving behavior.<sup>1</sup>

There appear to be a number of reasons for this condition. First, a variety of tasks has been used in problem solving research. The tasks found in the literature include such diverse problems as matchstick, Tower of Hanoi, jigsaw puzzles, anagram problems, concept identification problems, arithmetic computation problems, and standard mathematics textbook word problems. Also, problem solving research has been conducted by experimenters with quite different positions on the nature of problem solving. The traditional cognitive-Gestalt approach of such psychologists as Wertheimer (1959), Maier (1970), and Duncker (1945) is quite different from the associative learning theory approach characterized by the work of Maltzman (1955) and the Kendlers (Kendler & Kendler, 1962). More recently, especially within the past fifteen years, considerable effort has been devoted to the development of an information processing approach to the study of problem solving. The well-known work of Newell and Simon (1972) is representative of the information processing view of the problem solving process. Thus, although much exciting and potentially fruitful work is being conducted by psychologists, very few definitive answers to the questions concerning the nature of learning and instruction in mathematical problem solving are available at the present time. It is likely that these answers will result only from several years of intensive study that reflects a cooperative effort by mathematics educators, psychologists, and classroom teachers.

### Overview of This Paper

The intent of this paper is to describe the philosophy and activities of the Mathematical Problem Solving Project (MPSP) at Indiana University. The paper will contain four main sections:

1. the critical issues and questions related to mathematical problem solving,
2. nature of the MPSP,
3. thrust of the work of MPSP at Indiana University, and
4. plans for future research.

The main focus of this paper is on the research and development efforts underway at Indiana University. Included in this effort is a serious

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<sup>1</sup>Kilpatrick (1969) suggests that serious attention to problem solving by mathematics educators has developed primarily within the last ten or so years.

attempt to develop a conceptual framework for mathematical problem solving. The development of such a framework will center on the creation of a model for mathematical problem solving. Since the creation of such a model is considered to be of utmost importance in developing a framework for future research and development efforts, an extensive discussion of models of problem solving is included. It is hoped that the positions posed and the efforts described will stimulate valuable discussion concerning the key issues related to mathematical problem solving in the elementary schools.

### Critical Issues and Questions Related to Mathematical Problem Solving

The opening sentence of this paper stated that the development of children's problem solving abilities is a major goal of elementary school mathematics. It is interesting that while few educators would disagree with this claim there is little evidence that a serious attempt is being made to attain this goal. No single factor can be identified as causing this state of affairs to exist. Instead the problem can be attributed to a number of causes. The following are among the most prominent:

1. Problem solving is the most complex of all intellectual activities; consequently, it is the most difficult intellectual ability to develop.
2. Elementary school mathematics textbooks typically are deleterious rather than facilitative in developing problem solving skills and processes in children.
3. Elementary school teachers do not view problem solving as a key feature of their mathematics programs.

Before suggestions are presented for remedying the present situation it is appropriate to elaborate on causes 2 and 3.

It is the author's opinion that the overwhelming majority of the activities presented in elementary mathematics texts as problems are actually little more than exercises designed for practicing the use of a formula or algorithm. A second criticism is that textbooks do not include enough situations which involve real-world<sup>2</sup> applications of mathematics.

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<sup>2</sup>The term "real-world" is difficult to define since a real-world or real-life problem for one person may not be a real-life problem for another. Although interest rate and grocery shopping problems are very real in the sense that such problems are encountered daily by adults, they are often not even problems for children because children are not interested in them.

The third cause is the result of several factors. It is a fact that most elementary school teachers perceive mathematics to be a static and closed field of study. To them mathematics is more mechanics than ideas, and involves very little independent or original thought. Of course elementary teachers cannot be blamed for their perception of mathematics since it is based primarily on educational experiences which stressed memorization of rules, formulas, and facts. However, the view of mathematics which is held by elementary teachers is a part of a vicious cycle which has developed. Children are not learning to become good problem solvers because their mathematics textbooks do not provide appropriate opportunities for them to solve problems and because their teachers do not view problem solving as important. At the same time, teachers do not view problem solving as important because it was not given priority status when they studied mathematics. This condition cannot be rectified by attempting to convince preservice teachers of the importance of problem solving. At Indiana University preservice elementary school teachers are required to take nine semester hours of mathematics and three semester hours of methods of teaching mathematics. Even this uncommonly good situation does not allow sufficient time to overcome ten or more years of "bad" experiences with mathematics. Also, young teachers are prone to model their teaching behavior after the behavior of their supervising teachers. Consequently, if little or no provision is made for developing children's problem solving skills by a student teacher's supervising teacher, it is unlikely that the student teacher will consider problem solving as an important part of the mathematics program.

Remedies for the existing conditions cannot ignore the need to improve current teacher training programs, but improved teacher training is only a small part of the solution. Even if teachers can be trained to view mathematics as an area accessible through experimentation and independent thought, they will probably resort to using whatever written materials are available in the classroom and these materials are, for the most part, not conducive to enhancing the development of problem solving abilities. Thus, serious and extensive efforts must begin to develop exemplary instructional materials in mathematics which have problem solving as their main focus. The Mathematical Problem Solving Project (MPSP), which will be described later, is attempting to satisfy the need for such problem solving materials by producing a series of modules devoted to the development of certain problem solving techniques and by collecting and categorizing problems suitable for use in the intermediate grades.

Attempts to develop instructional materials of any type must involve considerable reflection about the most important aspects of the topic being considered. In the course of developing modules which will teach children fundamental skills and processes of problem solving the following questions are among those which should be considered.



1. What kind and how much direction should be given in a module?
2. What instructional format is best suited to teaching children how to solve problems?

Of course, these are important questions, but they are not specifically related to mathematical problem solving. Instead, they are questions which are raised by writers of any sort in instructional materials. It is premature to attempt to answer these questions until answers to several more basic questions are found. Unfortunately, the knowledge that exists about how children solve problems and how problem solving should be taught is very limited. For example, no confident answers have been found for the most basic questions such as:

1. What prerequisite skills, abilities, etc. must children have to solve particular kinds of problems?
2. What aspects of the problem solving process can be taught to intermediate grade children?
  - a. Can children use various problem solving strategies effectively?
  - b. Can children learn to coordinate the cognitive processes which are needed in solving complex problems?

Clearly the answers to these questions to a certain extent must be based upon the intuition and experience of the persons involved in writing the materials. However, it is equally as important that these questions be attacked by considering the theoretical and research base underlying the various views toward teaching problem solving. It would be most unfortunate to have another curriculum project which devotes all its energies to the development of materials to the exclusion of attempting to further the scientific knowledge regarding learning and instruction in mathematical problem solving.<sup>3</sup>

The issues raised thus far have been concerned primarily with the role of problem solving in the existing mathematics curriculum and the development of instructional materials. Before these issues can be dealt with in an appropriate way it is essential that several more fundamental issues and questions be considered. These issues include the four previously mentioned and are listed with some discussion following.

1. Can problem solving be taught?

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<sup>3</sup>This view is also held by Richard Shumway and is presented in a position paper prepared by him for the MPSP (1974).

2. If problem solving can be "taught,"<sup>4</sup> what type of experiences most enhance the development of this ability?
3. What are the specific characteristics of successful problem solvers?
4. What prerequisite skills, abilities, etc. and what level of cognitive development must a student have in order to solve a particular class of problems?
5. Educators and psychologists generally agree that there are several factors which influence problem difficulty. What are the primary determinants of mathematical problem difficulty for children in grades 4-6?
6. There are several motivation factors which influence children's ability and willingness to solve mathematical problems. For example:
  - a. What types of problems are interesting to children in grades 4-6?
  - b. To what extent does a child's cognitive and emotional style influence her/his willingness to solve problems?
7. What problem solving strategies can children (grades 4-6) learn to use effectively? More fundamentally, can problem solving strategies be taught which are generalizable to a class of problems?
8. Since problem solving is also important in nonmathematical areas, the question arises concerning the extent to which learning to solve various types of mathematical problems transfers to solving nonmathematical problems (the issue is just as important if modified to read ". . . transfers to solving other types of mathematical problems").
9. There are a number of issues related to the method of instruction. Among the most important are:
  - a. Is the small group mode of instruction a better mode than either the large group mode or individual instruction in terms of teaching problem solving?

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<sup>4</sup>"Taught" is being used here in the sense that teaching can be viewed as facilitating the understanding of or knowledge about something. It does not imply necessarily direct intervention in the student's learning process.

- b. What aspects of the problem solving process should influence the choice of method of instruction? For example, should the type of problem solving strategy appropriate for a problem affect the instructional mode used?
  - c. The specific role of the teacher in problem solving instruction is an open issue. Are there certain aspects of the problem solving process which suggest a more directive role by the teacher than others?
  - d. How should problem solving instruction be organized and sequenced? For example, should specific skills (e.g., making tables) be developed before attention is directed toward teaching a particular strategy? To what extent should a hierarchy (in the sense of Gagne 1970) be followed in planning instruction in problem solving?
10. How do such characteristics of problems as difficulty, interest, setting, strategy, and mathematical content relate to one another?
11. Several models of the problem solving process have been suggested. Do any of these models adequately describe mathematical problem solving? Is there a need for developing a model for instruction in problem solving? An instructional model might be fundamentally different from a model of the solution process.

#### Specific Questions Under Study by the MPSP

The MPSP at Indiana University has selected several of these issues and questions for study: namely, Nos. 1, 5, 6(a), 7, and 11. Since these questions and issues have been given some careful thought, it is appropriate to discuss them briefly.

Question 1. Can problem solving be taught? Clearly, this is the most important question of all. Kilpatrick's (1969) review of mathematical problem solving indicated that very little research has been done regarding the influence of instruction on problem solving ability. The answer to this question probably will not be determined until more is known about the nature of solving problems and the relationships among the many factors which influence mathematical problem solving.

Question 5. What are the primary determinants of mathematical problem difficulty for children in grades 4-6? Psychologists generally focus on four main areas for investigating problem difficulty: (a) type of problem task; (b) method of presentation of the problem; (c) familiarity of the problem solver with acceptable solution procedures (strategies, skills, etc.); (d) problem size (e.g., a problem with several

dimensions, both relevant or irrelevant, is more difficult than a problem having fewer dimensions). Each of these areas has direct relevance for elementary school mathematical problem solving. Clearly, not all types of problems are appropriate for children of this age. What is less clear is the best method of presenting particular classes of problems to children. Language factors, complexity of the problem statement, role of concrete and visual materials, child's prior experiences, and type of problem are among the several factors determining the most appropriate method of presentation. Much valuable information could be gained by posing problems to students in different forms and versions and under varying conditions.

That the student's familiarity with acceptable solution procedures is an important determinant of problem difficulty raises a number of questions which must be considered.

1. Which skills and strategies are most important for aiding problem solving in mathematics in grades 4-6?
2. Which skills and strategies should be taught first?
3. Which, if any, strategies do students use naturally?
4. Which skills and strategies can be taught efficiently and effectively? Can any be taught?
5. Should the skills (e.g., making a table) be developed before concentrating on teaching a strategy (e.g., pattern finding), or should they be developed as the strategy is taught?
6. Does teaching a particular strategy really improve problem solving ability in the sense that for any problem a student will be able to choose the most appropriate strategy to use?

More questions are being raised than answers in this paper. This reflects the author's earlier statement that there are few definitive answers to the questions about learning and instruction in mathematical problem solving. The questions posed in the preceding paragraph are no exceptions. However, despite the lack of answers based on firm research evidence, there is considerable agreement that strategies can and should be taught. This claim will be discussed when Question 7 is considered.

Issues related to problem size and problem complexity are a major focus of the research efforts of the MPSP. Since these efforts will be discussed in the last section of this paper no more will be said about problem size in this section.

The four determinants of problem difficulty that have been discussed are certainly not the only ones. Rather, they are the ones to which psychologists have devoted the most attention. Maier (1970) stated that

there are several other important factors which make a problem difficult. In determining a list of causes of difficulty, he began with the assumption that there is no lack of knowledge on the student's part. Based upon this assumption he listed five potential causes of difficulty in addition to the four that have already been mentioned: (a) misleading incorrect solutions, (b) type of demands made upon idea-getting processes versus idea-evaluation processes, (c) difficulty in locating subgoals that can be reached, (d) lack of motivation, and (e) high degree of stress.

The factors which have been listed in the previous paragraphs illustrate the extreme complexity of problem solving. In addition psychologists have determined these factors primarily through highly controlled experimentation. In many of the "laboratory" studies there was no need to consider factors such as mathematical content, level of understanding of concepts, processes, and skills, and environmental influences since ability to perform the tasks used is not contingent upon these factors. Unfortunately, these factors are present in normal classroom instruction. Consequently, in addition to the determinants of problem difficulty which have already been mentioned, the teacher is confronted with the task of dealing with even more confounding factors in planning appropriate mathematical problem solving activities.

Question 6 (a). What types of problems are interesting to children in grades 4-6? This question cannot be answered without considerable knowledge of a student's background, experiences, cognitive ability, and psychological makeup. There is substantial evidence that learning is enhanced when instruction is meaningful and relevant to the student. It is reasonable to expect that this is also the case in learning to solve problems. There are no hard-and-fast rules for determining if a particular problem is interesting, but there are some general rules-of-thumb which can guide problem selection.

1. Be sure the problem statement (if written) is easy for the student to read.
2. Use personal words and terms in the statement of the problem. Try to make the student feel like he is a part of the problem.
3. Although "real-world" problems are often difficult to find, such problems have a high motivational value. (Most of the "interesting" real-world problems are too sophisticated for the level of mathematical understanding which intermediate grade children have).
4. Encourage students to make up their own problems.
5. Do not place the student in a stressful situation. For example, insistence on getting a correct answer in a short period of time is a good way to kill enthusiasm for working a problem.

The MPSP is developing a problem bank for grades 4-6. One of the criteria for selecting a problem for inclusion in the bank is that it must be interesting to children. Interest will be determined through extensive interviewing and observing children as they solve problems.

Question 7. What problem solving strategies can children learn to use effectively? In papers prepared for the MPSP, Greenes (1974) and Seymour (1974) offer specific recommendations regarding skills and strategies which should be taught. Greenes not only listed several strategies which can be taught to children in grades 4-6 but also made suggestions for sequencing problem solving activities. The skills and strategies Greenes identified include: estimate or guess, simplify, conduct an experiment, make a diagram, make a table, construct a graph, write an equation, search for a pattern, construct a flowchart, partition the decision space, and deductive logic.

Seymour considers such skills as "making a table" and "constructing a graph" as valuable aids to mathematical problem solving but would probably classify such skills as substrategies because they are really tools for applying a strategy. The strategies he considers appropriate for the intermediate grades include: analogy, pattern recognition, deduction, trial and error, organized listing, working backwards, combined strategies, and usual strategies which are unique to a problem.

The belief of mathematics educators like Greenes, Seymour, and Polya (1957) that strategies can be taught should be given serious consideration. Most of our knowledge about learning and instruction is based on the experiences of teachers who have thought long and hard about ways to help children learn. Although little research has been done on the effectiveness of teaching problem solving strategies, the fact that several master teachers are convinced of the feasibility of teaching children the use of certain strategies should encourage teachers who are planning to include problem solving as a part of their mathematics program.

Question 11. Do any of the models of the problem solving process adequately describe mathematical problem solving? The primary purpose of a model is to describe the salient and essential characteristics of the process or phenomenon which is being modeled. Any model of the problem solving process should be evaluated on the basis on the extent to which it not only identifies the essential aspects of the process but also the extent to which stages and relationships among those stages are identified.

An investigation of this question has evoked considerable inquiry within the MPSP, and it is a major theme of this paper. A discussion of models of mathematical problem solving is included in a later section of this paper.

### The Nature of the Mathematical Problem Solving Project

The Mathematical Problem Solving Project (MPSP), which is cosponsored by the National Council of Teachers of Mathematics and the Mathematics Education Development Center at Indiana University and funded by the National Science Foundation, is working toward the development of mathematical problem solving modules which can be inserted into existing curriculum of grades 4-6. Many types of problem situations will be included in these modules: real-world applications of mathematics (i.e., "real world" as the student sees it), problems related to the mathematics studies in the standard curriculum, mathematical recreations, and problems involving various strategies such as guess and test and pattern finding. While the MPSP is primarily a development project the materials being developed will be based upon research into the teaching and learning of problem solving and will be pilot tested in a number of elementary schools.

The project is in operation at three different centers: the University of Northern Iowa, the Oakland Schools (Pontiac, Michigan), and Indiana University. While the project has identified the central goal as being the development of problem solving modules for use in grades 4-6, each center plays a distinct role.

### The Role of the University of Northern Iowa (U.N.I.) Center<sup>5</sup>

The MPSP site at the University of Northern Iowa is directed by George Immerzeel. The primary role of the site is to develop a series of "skills"<sup>6</sup> booklets and associated problem solving experiences. Specifically, the center at U.N.I. is identifying the spectrum of required skills that are not part of the present curriculum and writing materials that build this spectrum for particular problem solving strategies.

After considering an extensive list of required problem solving skills and classifying these skills into those that are simple (require a limited set of tactics) and complex (requiring a variety of tactics), seven were identified as appropriate for students in grades 4 through 6:

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<sup>5</sup>This description summarizes the role of U.N.I. as reported by George Immerzeel and his associates.

<sup>6</sup>There is a semantics problem in trying to communicate ideas about problem solving. Terms like "skill," "strategy," "heuristics," and "techniques" connote different things to different people. The word "skill," as used by the University of Northern Iowa staff, refers to generic problem solving techniques which are needed in order to use a particular strategy. Thus, "making a table" is a skill, whereas "pattern finding" is a strategy.

1. using an equation,
2. using a table,
3. using resources (reading, formulas, dictionaries, encyclopedias),
4. using a model (physical model, graph, picture, diagram),
5. make a simpler problem,
6. guess and test, and
7. compute to solve.

Each of these skills is simple in that they involve a single principle tactic. They do not depend upon an interrelation among tactics as is the case in strategies such as pattern finding and goal stacking.<sup>7</sup>

A "skills booklet" will be written for each of the seven skills. These booklets will be designed to teach the subskills needed to use a particular skill. For example, for the Guess and Test Skills Booklet, approximately 100 problems were written and the skills necessary to solve the problems were identified. These skills were then incorporated into the booklet.

The skills booklet is written so that a student can use the booklet independent of teacher input and also so the teacher can use the booklet in a regular classroom setting. After completing each booklet, the student is given an evaluation that not only determines the student's success in the skills but is a guide to group placement for the problem solving experiences designed for the skills.

The problem solving experiences consist of a set of cards for each skill. These cards represent five levels of difficulty and a variety of interests. Although a majority of the problems are supposed to have a "real world" setting, there are also examples from all aspects of the curriculum. From this set of problems each student should be able to find problems that not only fit her/his interests but also are at a level of difficulty where the student will be challenged but have a reasonable chance for success. Also included in the problem set are problems in which the use of the mini-calculator is appropriate. These problems are identified so the student knows the calculator is suggested for the problem. A separate skills booklet for the mini-calculator will be developed which can be used with any type of problem solving strategy.

As the skills booklets and problem sets are developed, they will be field tested with students in grades 4-6 in the Malcolm Price Laboratory School of the University of Northern Iowa.

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<sup>7</sup> See Newell and Simon (1972) and Wickelgren (1974) for a description of goal stacking.



### The Role of the Oakland Schools Center<sup>8</sup>

David Wells is the director of the Oakland Schools Center. This center is responsible for preparing teachers to field test and help develop materials. The teachers will use their classrooms to field test the materials developed at Oakland, U.N.I., and Indiana University. Thus the Oakland Schools center operates the major field testing components of the project. Currently, there are twelve teachers participating actively in solving problems, discussing problem difficulty, identifying problem solving strategies, developing problems for use in modules, and contributing to the development of modules.

The participation of classroom teachers is an essential part of the project. It is also essential that these teachers teach in a school system which offers diverse socio-economic groupings of children. The Oakland Schools Center is ideally suited in this respect since it has approximately 260,000 students and 14,000 teachers and contains industrialized centers, suburban communities, and rural areas.

### The Role of Indiana University (I.U.) Center

The Mathematics Education Development Center, under the direction of John LeBlanc, is the third site involved in MPSP. The role of the I.U. center is twofold. First, it is involved in the development of one or more modules based on information gathered through work with individual and small groups of students. Second, the center has major responsibility for evaluating the materials developed at the other centers and for making suggestions for revision. At the same time, the staff of the Mathematics Education Development Center is best qualified among the three centers to conduct developmental research into the questions which will arise inevitably as the modules and problems are being created. To date, research problems have been identified relate to problem difficulty and complexity and techniques for observing and interviewing children as they attempt to solve problems. The thrust of the work of the I.U. center will be discussed in more detail in a later section.

### Interrelationships of the MPSP Centers

The roles of the three centers have been described briefly but the interrelationships among the centers has not been specified. Interaction

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<sup>8</sup>This description summarizes the role of the Oakland Schools as reported by Stuart Choate, Assistant Director of the Oakland Schools Center.

among the centers is determined on the basis of need for reaction to ideas being investigated and materials being developed. For example, it is expected that materials devised by one center will be reacted to by the other centers. In this respect, there is a cyclic pattern of continual development, testing, and evaluation of materials which are produced (see Figure 1). Also, all three centers will be involved in identifying researchable issues for close scrutiny by the I.U. center.

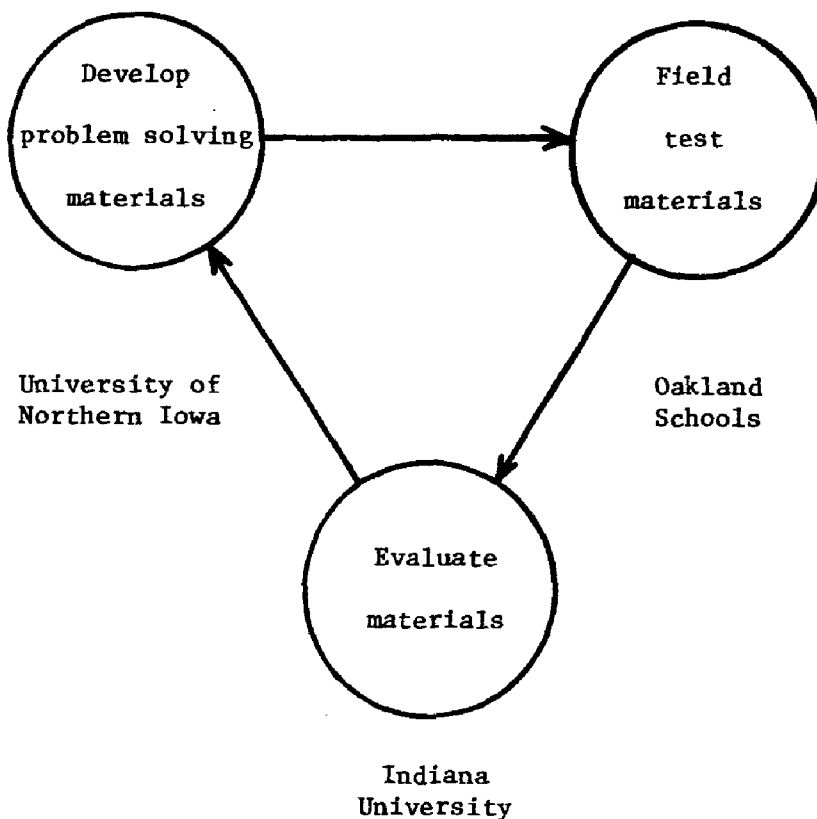


Figure 1. Interrelationship of primary roles of the MPSP centers.

A final word should be said regarding the feasibility of a tri-site project. Such an organizational structure necessitates some confusion, inefficiency, and duplication of efforts that must be taken into account in assessing the project. However, despite these shortcomings the tri-site aspect is viewed as a strength rather than a weakness of the project. The collaboration of educators with different

interests, experience, and expertise has been proposed by several leading curriculum developers. Having three centers offers a broader base for disseminating the materials which will be developed and provides a wider range of expertise in the areas of teaching, materials development, evaluation, and research.<sup>9</sup>

#### Focus of Efforts in the MPSP at Indiana University

This section is devoted to a description of the research and development work at Indiana University during 1974-75. Also, the current status of the model of mathematical problem solving which is evolving will be discussed. Although the development of a model has been given little direct attention during the past year, it seems appropriate to present it in this paper in order to elicit the reader's reactions.

The work of the I.U. center during the past year focused primarily on intensive observation of students' problem solving behavior, the development of a problem bank, and the creation of a problem solving module. The details of each of these three aspects of this work are discussed in the sections which follow.

#### Observation of Fifth-Grade Students

In order to get a better feeling for what types of problems students find interesting and to investigate if students employ any discernible strategies as they solve problems, the decision was made to spend some time (approximately 6 weeks) observing fifth-grade children as they attempted to solve problems without having any prior instruction. Fifth-graders were used because it seemed reasonable to fix the age level of the children so that developmental factors related to age would not have to be dealt with.

Approximately eighty problems were found that were suitable for most fifth-graders. The problems were selected on the basis of: relevance to fifth-grade mathematics, potential interest for fifth-graders, and "nonroutineness" (i.e., problems that are not standard textbook "story problem"). Consideration also was given to selecting problems which could be solved in more than one way. Ten of the problems were selected for use in interviewing students.

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<sup>9</sup>This view was articulated by James Gray who is the N.C.T.M. representative on the MPSP Advisory Board.

Two classes comprising approximately sixty fifth-grade students were interviewed as they attempted to solve some of the ten problems. The first class of students was interviewed individually and in groups of two, three, and four as they worked a set of four problems. When groups of students were interviewed, it proved too difficult to identify from audio recordings the processes used by individuals. Thus, all students in the second class were interviewed individually. The findings from the interviews were:

1. Very few of the students wrote anything down. Some drew a figure, but only after it was suggested by the interviewer.
2. Most students had difficulty retaining multiple conditions and considering two or more conditions at the same time.
3. Students often solved a problem that was not the stated problem. They misread the problem or misinterpreted the problem.
4. Students in general did not use strategies, although a few attempted to identify patterns for some problems.

The observation that many students were unable to coordinate multiple conditions in a problem (finding 2) deserves elaboration. One of the problems presented to students was the following:

There are 5 cups on the table. John has 9 marbles, and he wants to put a different number of marbles under each cup. Can he do this? Explain.

There are three different conditions to coordinate: five cups, nine marbles, and a different number of marbles under each cup. (Of course, "John" cannot perform this task.) Some students ignored the third requirement and came up with 2, 2, 2, 2, 1 as their answer. Other students ignored the condition of having nine marbles and arrived at 4, 3, 2, 1, 0 for an answer. It should be pointed out that although many students did not initially coordinate all of the conditions, they were able to do so after rereading the problem or being given a simple clue by the interviewer. It should be added that it is possible that students did not use all of the conditions because they would not have found a way to put the marbles under the cups otherwise. It is likely that they have been conditioned to find an "acceptable" answer at all costs. To them, getting an answer is the most important thing; getting an answer that makes sense is something else. This situation is probably not the fault of the students but the fault of a society which stresses immediate results and values quantity more than quality.

Mini-Instruction of fifth-grade. The results of the interviews suggested that although the students were unsuccessful for a variety of reasons, they did benefit from the questions asked and the hints given

by the interviewer. Thus, it seemed feasible to devise short sequences of instructional activities which would focus on helping children in the areas that appeared to cause them the most trouble.

A fifth-grade class, different from those interviewed, was divided into four groups (3 groups of 8 children and 1 group of 7 children). The groups were approximately equal in ability based on the scores from a pretest on mathematical reasoning. Each group was given forty-five minute of instruction on each of four consecutive days. The instruction varied among groups by what was stressed. The four different instructional stresses were based on the findings from the interviews. They were:

1. Using Strategies: This group worked on using "pattern finding" and "simplification" in solving problems.
2. Coordinating Conditions: This group considered the conditions of the problems and checked that the solution satisfied all of the conditions.
3. Understanding the Problem: This group was given ways to help understand what a problem is asking such as drawing a figure or distinguishing between relevant and irrelevant information.
4. Working Problems: This group was given no particular instruction. The students were given the problems and asked to work them. They were told if they had the solutions right or wrong and given hints when necessary.

Each group was given nearly the same set of problems over the four-day period. These problems were selected because they were appropriate for instruction in each group. At the end of the four-day instructional period, a posttest of four problems was given to all the students to see if any change in their problem solving behavior had occurred. In addition, two students from each group were individually interviewed as they worked the posttest.

There was no attempt to compare the groups statistically in terms of problem solving performance. This was not an experimental study to determine which of four instructional techniques was the best, but rather an exploratory investigation of the feasibility of providing instruction in very specific aspects of the problem solving process. As this point the primary interest was to try out ideas in order to gain a narrower focus,

not to conduct careful planned and controlled experiments to test well-formed hypotheses.

The results of the mini-instruction were inconclusive. Although the group which received instruction on using strategies seemed to benefit the most from the instruction, the teacher variable may well have been the factor that caused this to happen since each group had a different teacher. In general, the extent of the influence of the small group instructional sessions is unclear. However, the insight gained into the behavior of fifth-graders in small group problem solving situations was invaluable. Interviewing and observing students as they work on mathematical problems has continued to be a primary activity at the I.U. center.

#### Development of a Problem Bank and Problem Categorization Scheme

The second major thrust of the I.U. center has been toward the development of a large bank of problems of a wide variety of types. As the size of this bank has grown, it has become necessary to determine a scheme for categorizing the problems so that retrieval of problems will be efficient. A substantial effort has been undertaken to devise a suitable categorization scheme. In pursuit of this scheme the purposes of having a problem bank had to be clarified. The purposes of the problem bank are:

1. to provide classroom teachers with a source of problems of various types, and
2. to have available a wide range of problems with respect to structure and mathematical complexity, mathematical content, problem setting, strategies used in solving the problems, interest, etc. for use in development of problem solving materials.

One important use of the problem bank is as a source of problems exemplifying a particular strategy. For example, if a teacher wishes to illustrate the use of the "pattern finding" strategy, he/she can go to the problem bank and choose problems designated as "pattern finding" problems.

In order to categorize the problems in the bank four dimensions were identified: the setting of the problem, the complexity of the problem, strategies applicable for a problem, and the mathematical content of the problem. Initial attempts to sort out the components of each category resulted in the following outline for a categorization scheme.

- I. The setting of problems
  - A. Verbal setting
    1. Simple statement
    2. Statement in story form
    3. Statement in game form
    4. Statement in project form
  - B. Auxiliary nonverbal setting (a verbal setting accompanied by nonverbal information or materials which are not essential to solving the problem)
    1. Diagram/picture/graph
    2. Concrete objects
    3. Acting out the problem
    4. Hand calculators and other "facilitative" devices
  - C. Essential nonverbal setting (nonverbal information or materials essential to solving the problem)
    1. Diagram/picture/graph
    2. Concrete objects
    3. Acting out the problem
    4. Hand calculators and other "facilitative" devices
- II. Complexity of problems
  - A. Complexity of the problem setting
    1. Number of words
    2. Number of conditions (numerical and nonnumerical)
    3. Type of connectives among conditions
    4. Familiarity of setting
    5. Amount of superfluous information
    6. Number of clues provided (verbal and nonverbal)
  - B. Complexity of the solution process
    1. Familiarity with the type of solution
    2. Number of questions posed
    3. Type of connectives among questions
    4. Number of variables
    5. Type of connectives among variables
    6. Number of different operations required
    7. Type of operations required
    8. Number of steps required to reach solution
- III. Problem solving strategies
  - A. Pattern finding
  - B. Systematization
  - C. Visual perception
  - D. Inference
  - E. Trial-and-Error
  - F. Use and/or development of visual aids
  - G. Use and/or development of simpler problems
  - H. Recall and use of previous experiences

#### IV. Mathematical content

Since the problem bank will be used within the structure of the existing mathematics curriculum, the components of this category should be determined on the basis of topics included in various grade-five mathematics textbooks.

Problems which exemplify the use of various strategies have not been difficult to find. Carole Greenes and Dale Seymour have provided the MPSP with large collections of excellent problems which illustrate particular strategies and which are appropriate for use in the intermediate grades. Complexity has proven to be the most challenging category to consider. Several weeks of intensive study resulted in a revision of the outline related to the complexity of problems. The revised outline is presented here without discussion. Work is now underway to determine if factors included in this outline are critical in the determination of problem complexity.

##### I. Complexity of problem statement

###### A. Vocabulary

1. Word frequency
2. Specialized use

###### B. Sentence factors (conceptualization of phrases)

1. Number of simple sentences
2. Average number of words per sentence
3. Decodability of phrases

###### C. Amount of information

1. Numerals and symbols
2. Necessary numerical and nonnumerical data
3. Questions asked

###### D. Interest factor

1. Number of personal words
2. Number of concrete nonmathematical words

##### II. Complexity of the focusing process

###### A. Interrelationships of conditions

1. Number of bits of irrelevant data
2. Types of connectives between conditions (and, or, if . . . , then)
3. Order of presentation of the givens and/or operations
4. Logical structure of the problem

###### B. Interrelationships of goals

1. Leading questions
2. Corollary questions
3. Completely disjoint questions
4. Related questions



- III. Complexity of the solution process
  - A. Unique vs. non-unique vs. no solution
  - B. Mathematical content involved
  - C. Types of strategies that could be used effectively
  - D. Minimum number of subgoals
  - E. Types of goals
- IV. Complexity of evaluation
  - A. Ease of checking solution
  - B. Ease of generalizing solution

### Module Development

The development of instructional materials on pattern finding was begun. Pattern finding was chosen as the focus of the module because the students had an accurate understanding of the word "pattern" and used it in conversation. Also, there is a wealth of problems which involve pattern finding in their solutions. Preliminary versions of parts of the module have been tested in fifth grade classrooms. No formal evaluation of the extent to which students learn to use a pattern finding strategy has been conducted. Instead, the testing has concentrated on readability of the materials, clarity of presentation, format used, and interest level.

The attempt to develop a problem solving module on pattern finding and determine a scheme for categorizing mathematical problems necessitated a careful examination of the behaviors, both affective and cognitive, which are demonstrated as a student tries to solve a problem. This analysis involved an attempt to determine a model of the problem solving process which emphasizes the most important components of the process and provides an accurate description of how successful problem solvers think.

### Toward a Model of the Problem Solving Process

A search of the literature of problem solving revealed that several attempts have been made to devise a model which describes problem solving. It was appropriate to study some of these models in order to create a model which approximates the process for solving mathematical problems.

Dewey's model of reflective thinking. In his classic book, How we Think, Dewey proposes five phases of reflective thought (Dewey, 1933). While reflective thought is not synonymous with problem solving, it is clear that reflective thought is an essential part of problem solving. The five phases are: 1. suggestion, direct action upon a situation is inhibited thereby causing conscious awareness of being "in a hole" (p. 107); 2. intellectualization, an intellectualization of the felt difficulty leading to a definition of the problem; 3. hypothesizing, various hypotheses are identified to begin and guide observations in the collection of factual material; 4. reasoning, each hypothesis is mentally

elaborated upon through reasoning; and 5. testing the hypothesis by action, "some kind of testing by overt action to give experimental corroboration, or verification, of the conjectural idea" (pp. 113-4).

Dewey is careful to point out that these phases do not necessarily follow one another in any set order. This analysis is valuable in identifying stages in reflective thinking and thus, in problem solving. However, it considers only the logical aspects of reflective thought but does not consider nonlogical "playfulness" or intuition. It has been suggested (Getzels, 1964) that Dewey's formal steps are more a statement of one type of scientific method than an accurate description of how people think. As a result, this model of the process of solving problems may describe how students ought to think, but it does not describe how students usually do think when they are solving problems.

Johnson's model of problem solving. Whereas Dewey's model reflects a logical analysis of problem solving, Johnson (1955) has provided an analysis which is oriented to the psychological processes related to problem solving. Johnson's model is of particular interest because it provides a framework in which "to interpret measures of problem difficulty such as solution time" (cited in Bourne et al., 1971, p. 56). Three stages are included in his model:

1. preparation and orientation--the student gets an idea of what the problem involves;
2. production--the consideration of alternative approaches to a solution and the subsequent generation of possible solutions; and
3. judgment--the determination of the adequacy of a solution and the validity of the approach used to arrive at the solution.

In addition to providing information about problem difficulty this model offers a dimension that is not present in Dewey's model--it leads to speculation about the effects of instruction. In Johnson's model preproduction activity by the problem solver is just as important as the production stage. Unfortunately, little is known about the preparation state because researchers have preferred to investigate problem situations which are well-defined for the student. Thus, the preparation stage plays a less important role. Future research efforts should include studies which focus on the preparation stage of problem solving by examining problems for which the student is not fully prepared.

Polya's model of problem solving. George Polya's extensive writings have been a source of much valuable information regarding the problem of teaching problem solving in mathematics (Polya, 1957, 1962). Unlike Dewey and Johnson, Polya's concern lies primarily with mathematical problem

solving. To him, teaching problem solving involves considerable experience in solving problems and serious study of the solution process. The teacher who wants to enhance her/his student's ability to solve problems must direct the student's attention to certain key questions and suggestions which correspond to the mental operations used to solve problems. In order to group these questions in a convenient manner Polya suggests four phases in the solution process:

1. understanding the problem,
2. devising a plan,
3. carrying out the plan, and
4. looking back.

Since Polya's four phases are familiar to most mathematics educators interested in mathematical problem solving, no discussion of his model will be presented here. It should be pointed out that instead of being a description of how successful problem solvers think, his model is a proposal for teaching students how to solve problems. While this model may be valuable as a guide in organizing instruction in problem solving, it is too gross to be of much help in identifying potential areas of difficulty for students or clearly specifying the mental processes involved in successful problem solving.

Webb's model of problem solving. After reviewing the existing literature on mathematical problem solving, Webb (1974) devised a model which is purported to be a synthesis of the various models described in the literature. This model contains three main stages in solving a problem:

1. preparation--includes defining and understanding the problem; understanding what is unknown, what is given, and what the goals are;
2. production--includes the search for a path to attain the goals; recall of principles, facts, and rules from memory; generation of new concepts and rules to be used in solving the problem; and development of hypotheses and alternative plans that may lead to one or more goals; and
3. Evaluation--includes checking subgoals and the final solution; and checking the validity of procedures used during preparation and production.

Webb stated that his model "is not a hierarchial model in that preparation always comes before production which always must precede evaluation. This is more a cyclic model" (Webb, 1974, p. 4). The models of Polya and Webb have proven to be useful to the staff at the Indiana University center of the MPSP as rudimentary models from which a more detailed and refined model can be developed.

Some other models of problem solving. In addition to the models proposed by Dewey, Johnson, Polya and Webb, at least two other thoughtful models have been developed. The first is the model of Klausmeir and Goodwin (1966). The major aspects of their model are highlighted below without discussion:

1. setting a goal,
2. appraising the situation,
3. trying to attain the goal,
4. confirming or rejecting a solution, and
5. reaching the goal.

The major points of the second model by Wallas (1929) are also highlighted below without discussion:

1. preparation,
2. incubation (a mulling-over period),
3. illumination (the conception of a solution), and
4. verification,

#### A Working Model of Problem Solving for the MPSP at Indiana University

The primary limitation of each of the models that have been discussed is that they are either prescriptive (viz., Dewey and Polya) or only grossly descriptive (viz., Johnson, Klausmeir and Goodwin, Wallas, and Webb). The prescriptive models suggest techniques to help the student to be a better problem solver. The descriptive models may be more valuable in the sense that they identify phases the student goes through during problem solving. A goal of the MPSP is to devise a more detailed and refined descriptive model.

The search for such a model has led to an investigation of information processing approaches to problem solving research. With the possible exception of gestalt psychology, information processing theory seems to be the only psychological theory which has problem solving as a central focus. A primary thrust of information processing theory is to develop a description of specific types of problems that is precise enough to enable an explanation of problem solving behavior in terms of basic cognitive processes. The most complete description of information processing theory has been presented by Newell and Simon (1972). Wickelgren (1974) has attempted to develop an operationalized theory of problem solving by combining elements of information processing theory and the ideas of master teachers like George Polya.

The work of Newell and Simon, and Wickelgren has led the author to the model for solving mathematics problems which is described in the following paragraphs. This model is, of course, not as refined as it should be nor does it necessarily generalize to all types of successful mathematical problem solving behavior. However, it does pinpoint some critical components of problem solving behavior which are missing in the other models. Six distinct, but not necessarily disjoint, stages are included in this model:

1. problem awareness,
2. problem comprehension,
3. goal analysis,
4. plan development,
5. plan implementation, and
6. procedure and solution evaluation.

It should be emphasized that these stages are not necessarily sequential. In fact it only rarely happens that these stages do occur sequentially and distinctly from each other.

In keeping with an information processing approach to building a model, it would be desirable to devise a flow chart that would describe the student's cognitive processes as progress is made from Problem Awareness through Procedure and Solution Evaluation. However, since the stages are not hierarchically ordered or even distinct, for most problems it is not possible to devise a completely accurate diagram of the flow of progress during problem solving. Figure 3 (page 82) is a rough description of the way in which the stages of the model are related.

Stage 1: Problem awareness. A situation is posed for the student. Before this situation becomes a problem for the student, he/she must realize that a difficulty exists. A difficulty must exist in the sense that the student must recognize that the situation cannot be resolved readily. This recognition often follows from initial failure to attain a goal. This view of what constitutes a problem is consistent with Bourne's description of a problem situation as one in which initial attempts fail to accomplish some goal (Bourne et al., 1971). A second component of the awareness stage is the student's willingness to try to solve the problem. If the student either does not recognize a difficulty or is not willing to proceed in trying to solve the problem, it is meaningless to proceed (see Figure 2).

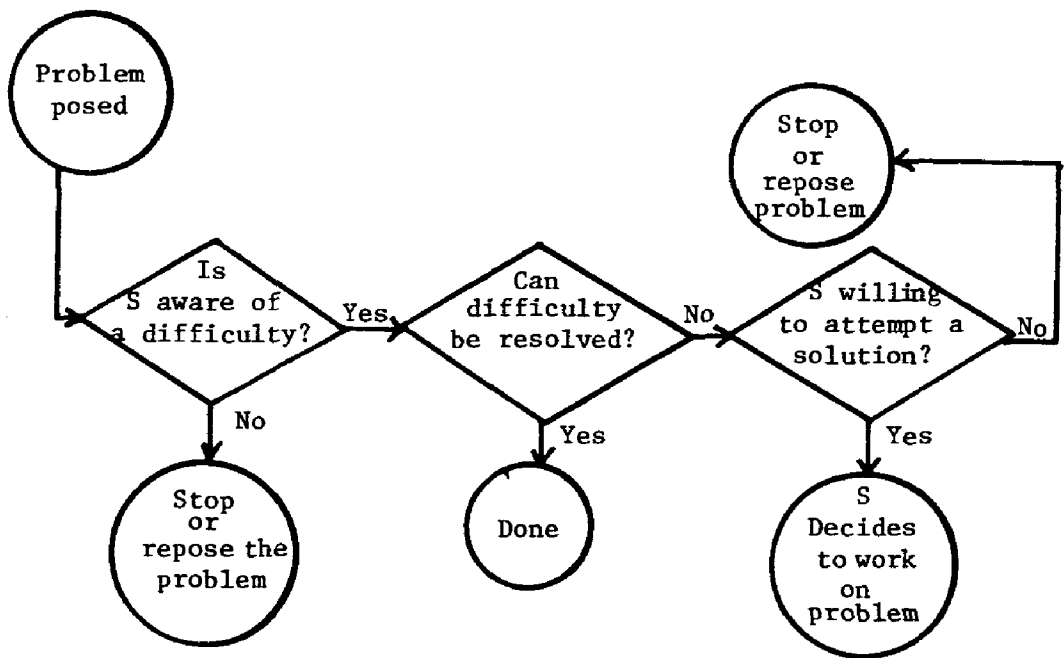


Figure 2: Schematic representation of problem awareness.

Stage 2: Problem comprehension. Once the student is aware of the problem situation and declares a willingness to eliminate it as a problem, the task of making sense out of the problem begins. This stage involves at least two sub-stages: translation and internalization. Translation involves interpretation of the information the problem provides into terms which have meaning for the student. Internalization requires that the problem solver sort out the relevant information and determine how this information interrelates. Most importantly, this stage results in the formation of some sort of internal representation of the problem within the problem solver. This representation may not be accurate at first (or it may never be accurate, hence the student fails to solve the problem), but it furnishes the student with a means of establishing goals or priorities for working on the problem. It is here that the nonsequential nature of the model shows up for the first time. The accuracy of the problem solver's internal representation may increase as progress is made toward a solution. Thus, the degree of problem comprehension will be a factor in several stages of the solution process.

Stage 3: Goal analysis. It seems that the problem solver may jump back and forth from this stage to another. For some problems it is appropriate to establish subgoals, for others subgoals are not needed. It is often true that the identification and subsequent attainment of a subgoal aids both problem comprehension and procedure development.

Goal analysis can be viewed as an attempt to reformulate the problem so that familiar strategies and techniques can be used. It may also involve an identification of the component parts of a problem. It is a process which moves from the goal itself backwards in order to separate the different components of the problem. Thus, goal analysis actually includes more than a simple specification of given information, specification of the relationships among the information, and specification of the operations which may be needed (see Resnick & Glaser, 1976, for a more detailed discussion of goal analysis).

Stage 4: Plan development. It is during this stage that the problem solver gives conscious attention to devising a plan of attack. Developing a plan involves much more than identifying potential strategies (e.g., pattern finding and solving a simpler related problem). It also includes ordering subgoals and specifying the operations which may be used. It is perhaps this stage more than any other that causes difficulty for students. It is common to hear mathematics students proclaim after watching their teacher work a problem: "How did he ever think of that? I never would have thought of that trick." The main sources of difficulty in learning how to formulate a plan of attack emanate from the fact that students are prone to give up if a task cannot be done easily. Of course, if problems can be done too easily, they are not really problems. A good problem causes initial failure which too often results in a refusal to continue. This state of affairs is not the fault of students, but rather the fault of teachers who do not recognize that initial failure is a necessary condition for problem solving (Shumway, 1974). It may also be true that students are unable to devise good plans because they have few plans at their disposal. There is preliminary evidence from work done at the Indiana University center of MPSP that many children in grades 4-6 proceed primarily in a trial-and-error fashion until they either find a "solution" that satisfies them or give up. Equipping students of this age with a few well-chosen strategies may facilitate their ability to plan.

Another source of difficulty for students at this stage is in ordering subgoals and specifying the operations to be used. For many students the hardest part of problem solving lies with knowing what to do first and organizing their ideas. Consequently, in addition to teaching students strategies, attention must be given to helping them organize their thinking and planning.

Stage 5: Plan implementation. At this stage, the problem solver tries out a plan which has been devised. The possibility that executive errors may arise confounds the situation at this stage. The student who correctly decides to make a table and look for a pattern may fail to see the pattern due to a simple computation error. Errors of this type probably cannot be eliminated, but they can be reduced if instruction on implementing a plan also considers the importance of evaluating the plan while it is being tried. Thus, while stages 5 and 6 are distinct, they are not disjoint. The main dangers of stage 5 are that the problem solvers may forget the plan, become confused as the plan is carried out,

or be unable to fit together the various parts of the plan. Fitting together the parts of a plan can be a very difficult task in itself. This difficulty may arise from the fact that the best sequencing of steps in the plan or the best ordering of subgoals may not be clear to the problem solver. For some problems the sequencing of subgoals does not matter, while for others it is essential that subgoals be achieved in a particular order. The reader is referred to Chapter 6 of Wickelgren's book How To Solve Problems for an in-depth analysis of techniques for defining subgoals and using them to solve problems (Wickelgren, 1974).

Stage 6: Procedures and solution evaluation. Successful problem solving usually is the result of systematic evaluation of the appropriateness of the decisions made during the problem solving and thoughtful examination of the results obtained. The role of evaluation in problem solving goes far beyond simply checking the answer to be sure that it makes sense. Instead, it is an ongoing process that starts as soon as the problem solver begins goal analysis and continues long after a solution has been found. Procedure and solution evaluation may be viewed as a process of seeking answers to certain questions as the problem solver works on a problem. Representative of the questions which should be asked by the problem solver at each stage are the following:

1. problem comprehension stage--What are the relevant and irrelevant data involved in the problem? Do I understand the relationships among the information given? Do I understand the meaning of all the terms that are involved?
2. goal analysis stage--Are there any subgoals which may help me achieve the goal? Can these subgoals be ordered? Is my ordering of subgoals correct? Have I correctly identified the conditions operating in the problem?
3. plan development stage--Is there more than one way to do this problem? Is there a best way? Have I ever solved a problem like this one before? Will the plan lead to the goal or a subgoal?
4. plan implementation stage--Am I using this strategy correctly? Is the ordering of the steps in my plan appropriate, or could I have used a different ordering?
5. solution evaluation stage--Is my solution generalizable? Does my solution satisfy all the conditions of the problem? What have I learned that will help me solve other problems?

Figure 3 attempts to illustrate the interrelationships that exist among the stages in the model. It also suggests how a student might proceed in solving a problem.



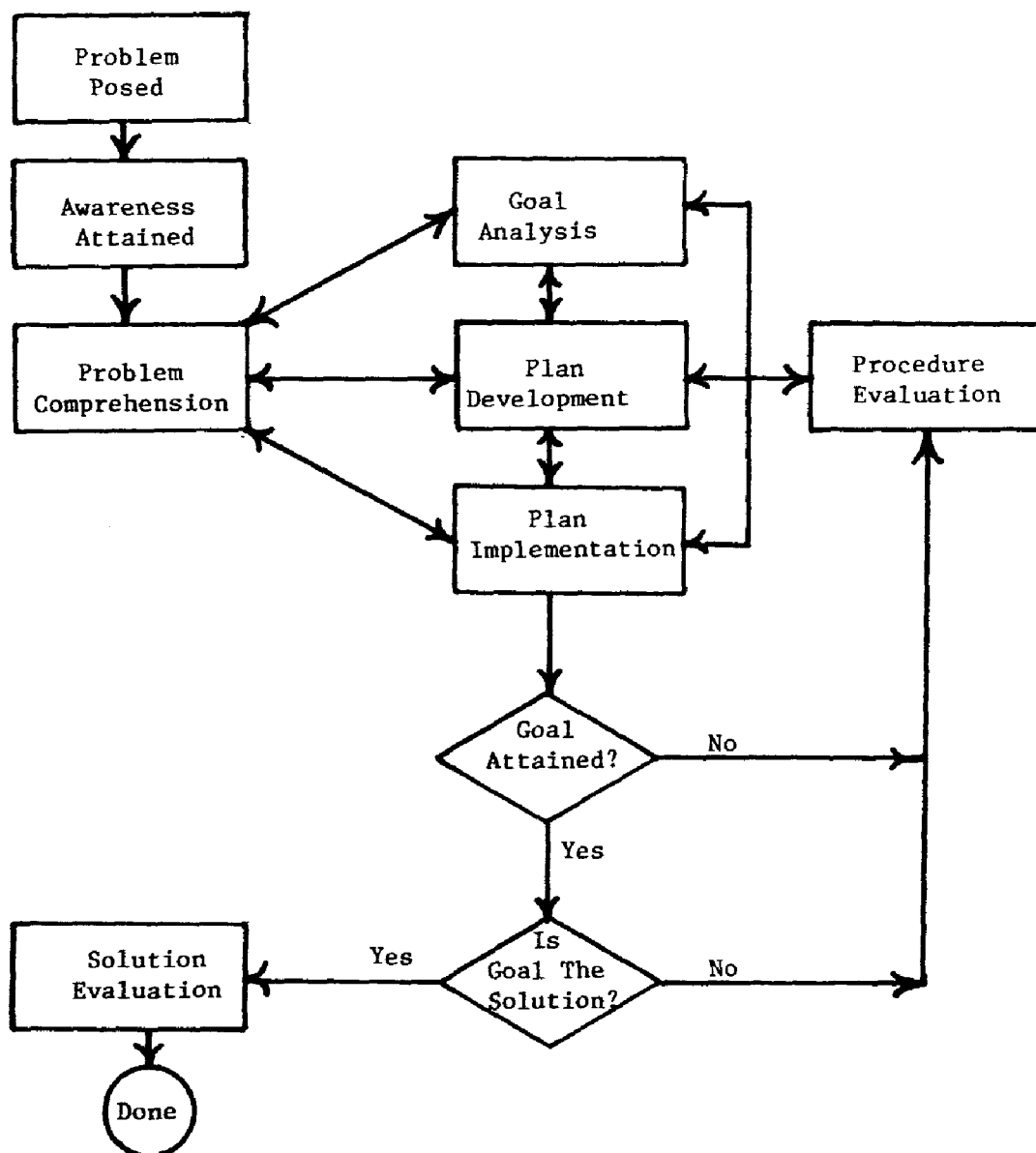


Figure 3: Schematic representation of a model of mathematical problem solving.

How the model may be used. The most valuable aspect of this model is that it provides a conceptual framework for identifying the factors which most influence success in problem solving. This framework can be useful to the teacher who is trying to organize appropriate problem solving experiences for students by highlighting various potential sources of difficulty for problem solvers. It also emphasizes that

teachers cannot be content to teach students how to solve problems by simply showing a few "tricks of the trade." Of course, the model does not describe problem solving for all types of problems and, in this sense at least, it is incomplete. But, it does supply a partial explication of a theory of problem solving which, although not fully conceptualized, is being created. The development of a theory of problem solving will give direction and add focus to any research efforts. Such a theory is needed critically within mathematics education at the present time. Many of the research efforts in mathematical problem solving which have been conducted were well-conceived and carefully done, but the results of these efforts have had little impact on instructional practice. This is partially due to the diversity of types of research and the conflicting results which have been obtained. It is also due to the fact that none of the results seem to be generalized to all types of mathematical problems. It may be that no single theory, and hence no single model, can accurately depict problem solving for all types of problems and all types of problem solvers. Even with the possibility of such a state of affairs, it is worthwhile to continue the search for a suitable model since such a search will provide valuable information about the nature of the problem solving process.

#### Plans For Future Research

Although the MPSP is primarily a development project, an investigation of a few research questions will be included as a part of the efforts during 1975-76. Much of the work done at I.U. during the past year can be classified as exploratory. Emphasis was placed on intensive observation of students, the collection of problems, the creation of a problem solving module, and the design of a suitable model for mathematical problem solving. While none of these endeavors can be considered research in the usual sense, all of the work at I.U. was conducted with a research spirit. That is, every effort was made to approach each issue in an open-minded and objective manner and to apply the scientific method of inquiry.

Perhaps the most valuable result of the work at the I.U. center was the identification of three areas within the problem solving process which cause difficulty for fifth-graders. Two of these difficulties are related to problem comprehension, while the third is related to plan development and implementation.

1. Students often misread or misinterpreted problems.
2. Students had difficulty retaining and coordinating multiple conditions in a problem.
3. Students do not appear to use any strategies during problem solving.

Further investigation of the first difficulty suggested that students often perceive a simplified version of a stated problem. The students then proceed to solve the problem as they perceive it. In a few cases, the students were not even aware that a problem existed. In other cases students had trouble understanding phrases in problems (e.g., "a checker in every row and in every column" and "every sixth night"). Clearly, students cannot solve problems they don't fully understand. It is important, then, to pay special attention to the factors which influence problem comprehension. More specifically, it is important to determine the primary determinants of reading difficulty since most mathematical problems are presented in a written form.

Several measures of comprehension of written passages have been developed by reading specialists. However, there is reason to believe that these measures may not be appropriate for written mathematical passages since mathematical English appears to be much different from ordinary English. Kane (1968) has suggested that there are at least four differences between mathematical English and ordinary English: (a) redundancies of letters, word, and syntax are different, (b) names of mathematical objects usually have a single denotation; (c) adjectives are more important in mathematical English than in ordinary English; and (d) the grammar and syntax of mathematical English are less flexible than in ordinary English.

If mathematical English is significantly different from ordinary English, it is essential that the nature of these differences be determined. Two members of the MPSP staff at I.U., Norman Webb and Barbara Moses, have designed a study which aims at identifying a reliable and accurate measure of comprehension of written mathematics problems. Their study will investigate the following questions:

1. Is the Cloze procedure<sup>10</sup> a reliable measure of comprehension for individual mathematical problems?
2. What is the relationship of certain stimulus measures of mathematical problem statements to the mean Cloze score percentage?
3. What stimulus measures are the best predictors of mean Cloze score percentage?

Stimulus measures will include such variables as the number of one-syllable words, nouns, personal words, symbols connectives, sentences, and clauses per 100 words as well as the number of words with specialized mathematical meanings and the average sentence length.

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<sup>10</sup> The Cloze procedure is a popular technique for measuring readability of long passages. The procedure involves deleting every nth word or symbol of a passage and replacing them with blanks. The student must fill in the blanks. The score is determined by the number of responses matching the deleted material. A high score indicates high readability.

Webb and Moses expect that one or two stimulus measures will be found that can be used to predict the difficulty of comprehending a mathematical problem. They also expect the Cloze procedure to prove to be an adequate measure of readability for mathematical problems. If such expectations are supported, the task of classifying problems according to complexity will be greatly reduced.

The fact that many of the fifth-graders were unable to coordinate and retain the conditions given in a problem has led to the design of a study to investigate particular issues related to this fact. Another MPSP staff member, Fadia Harik, has decided to explore the influence the number of conditions in a problem has on success in solving problems. In addition, she will investigate the effect certain types of teacher clues has on problem solving success. This aspect of her study arose from the observation that although fifth-graders do not initially coordinate multiple conditions simultaneously, they are able to do so in some problems if the teacher provides clues or asks the students to reread the problem.<sup>11</sup>

Research studies like those of Webb and Moses, and Harik have been carefully conceived, organized, and planned. Their questions have risen from a concern for developing a sensible theory of mathematical problem solving. It is only by conducting research based on a sound conceptual framework that any significant progress will be made toward developing instructional materials which will enhance children's ability to solve mathematical problems.

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<sup>11</sup> Both the study by Harik and the one by Webb and Moses have been completed since this paper was written. The interested reader can obtain information about the results of these studies by contacting the author of this paper.

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